

Decoupling inequalities and interlacement percolation on $G \times \mathbb{Z}$

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Abstract We study the percolative properties of random interlacements on $G \times \mathbb{Z}$, where G is a weighted graph satisfying certain sub-Gaussian estimates attached to the parameters $\alpha > 1$ and $2 \leq \beta \leq \alpha + 1$, describing the respective polynomial growths of the volume on G and of the time needed by the walk on G to move to a distance. We develop decoupling inequalities, which are a key tool in showing that the critical level u_* for the percolation of the vacant set of random interlacements is always finite in our set-up, and that it is positive when $\alpha \geq 1 + \beta/2$. We also obtain several stretched exponential controls both in the percolative and non-percolative phases of the model. Even in the case where $G = \mathbb{Z}^d$, $d \geq 2$, several of these results are new.

0 Introduction

Random interlacements offer a microscopic model for the structure left at appropriately chosen time scales by random walks on large recurrent graphs, which are locally transient. In this work we investigate the percolative properties of random interlacements on transient weighted graphs E of the form $G \times \mathbb{Z}$, where the walk on the weighted graph G satisfies certain sub-Gaussian estimates governed by two parameters $\alpha > 1$ and β in $[2, \alpha + 1]$, respectively reflecting the volume growth of G , and the diffusive or sub-diffusive nature of the walk on G . Random interlacements on the weighted graphs considered in this article occur for instance in the description of the microscopic structure left by random walks on discrete cylinders $G_N \times \mathbb{Z}$, with large finite bases

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G_N , which tend to look like G in the vicinity of certain points, when the walk on $G_N \times \mathbb{Z}$, runs for times comparable to the square of the number of points in G_N , see [22, 31]. In the spirit of [30], they are expected to occur in the description of the microscopic structure left by random walks on suitable sequences of large finite graphs E_N , which tend to look like E in the vicinity of certain points, when the walk runs for times proportional to the number of points in E_N .

Here, our main interest lies in the percolative properties of \mathcal{V}^u the vacant set at level u of random interlacements on E , as $u \geq 0$ varies. This set is the complement in E of the trace of the trajectories with labels at most u in the interlacement point process. These percolative properties are naturally related to various disconnection and fragmentation problems, see [1, 5–7, 9, 10, 23, 24, 28]. For instance in the case of random walks on the cylinders $(\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}$, with $d \geq 2$, one can construct couplings of the trace in a box of size $N^{1-\varepsilon}$ of the complement of the trajectory of the walk after completion of a suitable number of excursions to the vicinity of the box, with the vacant set of random interlacements in a box of size $N^{1-\varepsilon}$ in \mathbb{Z}^{d+1} , see [23, 24], and [4]. These couplings enable one to show that the disconnection time T_N of the cylinder by the walk has precise order N^{2d} . They also suggest a candidate limit distribution for T_N/N^{2d} as N goes to infinity, which brings into play the critical parameter u_* for the percolation of \mathcal{V}^u , see [23, 24]. Proving that such a limit holds presently rests on being able to sharpen controls on the percolative properties of \mathcal{V}^u when u is fixed but possibly close to u_* . In the case of random walks on $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$, similar couplings between the trace left by the complement of the trajectory of the walks at time uN^d in boxes of size $N^{1-\varepsilon}$, and the trace of \mathcal{V}^u in boxes of size $N^{1-\varepsilon}$ in \mathbb{Z}^d can be constructed, cf. [28]. They enable one to show that for small u there typically is a giant component containing order N^d sites in the complement of the trajectory of the walk at time uN^d , which is unique, at least when $d \geq 5$, and that for large enough u there typically are only small components. The critical value u_* for the percolation of \mathcal{V}^u is moreover conjectured to be the threshold for this fragmentation problem, separating the two behaviors mentioned above. The proof of such a conjecture analogously rests on the improvement of controls on the percolative properties of \mathcal{V}^u for u fixed but possibly close to u_* . When $(\mathbb{Z}/N\mathbb{Z})^d$ is replaced by a large d -regular graph on N vertices, with $d \geq 3$, and random walk runs on the graph up to time uN , one can indeed show that when $u < u_*$, for large N there typically is a component in the complement of the trajectory with order N vertices, which is unique, whereas for $u > u_*$ all components are small. The relevant threshold for this fragmentation problem is now the critical value u_* for the percolation of the vacant set of random interlacements on the d -regular tree, (the local model for typical points of large random d -regular graphs), cf. [6, 7, 9].

We now describe our results in more detail, but refer to Sect. 1 for precise definitions. We consider graphs of the form $E = G \times \mathbb{Z}$, where G is an infinite connected graph of bounded degree, endowed with bounded and uniformly positive weights along its edges. We assume that G is α -Ahlfors regular for some $\alpha > 1$, that is, see also (1.7),

$$\begin{aligned} &\text{the volume of balls of radius } R \text{ in the graph-distance on } G \\ &\text{behaves as } R^\alpha, \text{ up to multiplicative constants.} \end{aligned} \quad (0.1)$$

We endow the product graph $E = G \times \mathbb{Z}$ with weights, which take the value 1 for “vertical edges” in the \mathbb{Z} -direction, and agree with the corresponding weights on G for “horizontal edges” in the G -direction. These weights naturally determine a random walk on E , which at each step jumps to one of its neighbors with a probability proportional to the weight of the edge leading to this neighbor. Due to (0.1) the random walk on E is in fact transient, see below (1.8), and we assume that the Green density $g(\cdot, \cdot)$ has a power decay of the following kind: there is a β in $[2, \alpha + 1]$, such that

$$c(d(x, x') \vee 1)^{-\nu} \leq g(x, x') \leq c'(d(x, x') \vee 1)^{-\nu}, \quad \text{for } x, x' \text{ in } E, \quad (0.2)$$

with $\nu = \alpha - \frac{\beta}{2}$ (a positive number due to the constraints on α, β), and $d(\cdot, \cdot)$ the distance on E defined by:

$$d(x, x') = \max(d_G(y, y'), |z - z'|^\frac{2}{\beta}), \quad \text{for } x = (y, z), x' = (y', z') \text{ in } E, \quad (0.3)$$

where $d_G(\cdot, \cdot)$ stands for the graph-distance on G .

In fact, with the help of [11, 12], the above assumptions can be restated in terms of the following sub-Gaussian estimates for the transition densities $p_n^G(y, y')$ of the walk determined by the weighted graph G , see Remark 1.1 (2):

$$\begin{aligned} \text{(i)} \quad &p_n^G(y, y') \leq c n^{-\frac{\alpha}{\beta}} \exp\{-c(d_G(y, y')^\beta/n)^{\frac{1}{\beta-1}}\}, \quad \text{for } n \geq 1, y, y' \text{ in } G, \\ \text{(ii)} \quad &p_n^G(y, y') + p_{n+1}^G(y, y') \geq c n^{-\frac{\alpha}{\beta}} \exp\{-c(d_G(y, y')^\beta/n)^{\frac{1}{\beta-1}}\}, \\ &\text{for } n \geq 1 \vee d_G(y, y'), y, y' \text{ in } G, \end{aligned} \quad (0.4)$$

and c refers to positive constants changing from place to place.

The classical example $G = \mathbb{Z}^d$, $d \geq 2$, endowed with the natural weight equal to 1 along all edges thus corresponds to $\alpha = d$, $\beta = 2$, whereas the case of G , the discrete skeleton of the Sierpinski gasket endowed with its natural weight, corresponds to $\alpha = \frac{\log 3}{\log 2}$, $\beta = \frac{\log 5}{\log 2} (> 2)$, see [3, 15]. We also refer to [2, 3, 11, 12, 14], for many more examples and for equivalent characterizations of (0.4).

Random interlacements on the transient weighted graph E consist in essence of a Poisson point process on the state space of doubly infinite trajectories on E modulo time-shift, which tend to infinity at positive and negative infinite times, see [26], Sect. 1, and [21], Remark 1.4. A non-negative parameter u plays the role of a multiplicative factor of the σ -finite intensity measure of this Poisson point process. In fact, one constructs “at once”, on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the whole family \mathcal{I}^u , $u \geq 0$, of random interlacements at level u . These random subsets of E are infinite when u is positive, and come as unions of the ranges of the trajectories modulo time-shift with label at most u , in the canonical Poisson cloud constructed on $(\Omega, \mathcal{A}, \mathbb{P})$. The law on $\{0, 1\}^E$ of the indicator function of \mathcal{I}^u is characterized by the identity:

$$\mathbb{P}[\mathcal{I}^u \cap K = \emptyset] = \exp\{-u \operatorname{cap}(K)\}, \quad \text{for all finite subsets } K \text{ of } E, \quad (0.5)$$

with $\operatorname{cap}(K)$ the capacity of K , see (1.21).

There is by now substantial evidence that the percolative properties of the vacant set $\mathcal{V}^u = E \setminus \mathcal{I}^u$ play an important role in understanding various disconnection and fragmentation problems for random walks on large recurrent approximations of E , see in particular [7, 23, 24, 28].

To further discuss these properties, we define for x in E and $u \geq 0$,

$$\eta(x, u) = \mathbb{P}[x \overset{\mathcal{V}^u}{\longleftrightarrow} \infty], \quad (0.6)$$

where the above notation refers to the event “ x belongs to an infinite connected component of \mathcal{V}^u ”. It is known from Corollary 3.2 of [26] that the critical value

$$u_* = \inf\{u \geq 0; \eta(x, u) = 0\} \in [0, \infty], \quad (0.7)$$

does not depend on the choice of x . It is an important question whether u_* is finite and positive. We show in Theorem 4.1 that \mathcal{V}^u does not percolate for large u , that is:

$$u_* < \infty. \quad (0.8)$$

When $\alpha \geq 1 + \frac{\beta}{2}$, i.e. $\nu \geq 1$ in (0.2), we show in Theorem 5.1 that for small $u > 0$, \mathcal{V}^u percolates in “half-planes” of $E = G \times \mathbb{Z}$ that are product of a semi-infinite geodesic path in G with \mathbb{Z} , and in particular that

$$u_* > 0, \quad \text{when } \nu \geq 1. \quad (0.9)$$

Some special cases of (0.8), (0.9) are known, for instance when E is \mathbb{Z}^{d+1} , $d \geq 2$, endowed with its natural weight, see [18, 21]. Also when $\nu \geq 6$, the methods of Sect. 4 of [26] can likely be adapted to prove (0.9). But Theorems 4.1 and 5.1 yield further information. If $B(x, r)$ denotes the closed ball with center x in E and radius $r \geq 0$ in the $d(\cdot, \cdot)$ -metric, see (0.3), and

$\partial_{\text{int}} B(x, r)$ stands for the set of points in $B(x, r)$ neighboring $B(x, r)^c$, we introduce:

$$u_{**} = \inf \left\{ u \geq 0; \lim_{L \rightarrow \infty} \sup_{x \in E} \mathbb{P}[B(x, L) \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B(x, 2L)] = 0 \right\} \in [0, \infty], \quad (0.10)$$

where the event under the probability refers to the existence of a nearest neighbor path in \mathcal{V}^u between $B(x, L)$ and $\partial_{\text{int}} B(x, 2L)$. We trivially have $u_* \leq u_{**} \leq \infty$, and Theorem 4.1 actually shows that

$$u_{**} < \infty, \quad (0.11)$$

and that for $u > u_{**}$, the connectivity function has a stretched exponential decay

$$\mathbb{P}[x \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B(x, L)] \leq c e^{-c' L^\gamma}, \quad (0.12)$$

where $c(u)$, $c'(u) > 0$ and $0 < \gamma(u) < 1$ are constants possibly depending on u . We also introduce the value

$$\tilde{u} = \inf \left\{ u \geq 0; \lim_{L \rightarrow \infty} \sup_{\mathcal{P}, x \in \mathcal{P}} \mathbb{P}[B(x, L) \xleftrightarrow{* \mathcal{I}^u \cap \mathcal{P}} \partial_{\text{int}} B(x, 2L)] > 0 \right\}, \quad (0.13)$$

where \mathcal{P} runs over all the half-planes in E , and the event under the probability refers to the existence of a $*$ -path in $\mathcal{I}^u \cap \mathcal{P}$ between $B(x, L)$ and $\partial_{\text{int}} B(x, 2L)$, see below (3.8) for the definition of a $*$ -path in \mathcal{P} . We show in Theorem 5.1 and Corollary 5.5 that

$$\tilde{u} \leq u_*, \quad (0.14)$$

that for $u < \tilde{u}$ and for any half-plane \mathcal{P} , and x in \mathcal{P} , as above,

$$\begin{aligned} & \mathbb{P}[\text{the component of } x \text{ in } \mathcal{V}^u \cap \mathcal{P} \text{ is finite and intersects } \partial_{\text{int}} B(x, L)] \\ & \leq c e^{-c' L^{\gamma'}}, \end{aligned} \quad (0.15)$$

with $c(u)$, $c'(u) > 0$ and $0 < \gamma(u)' < 1$, constants possibly depending on u , that in (0.13) the quantity under the \liminf satisfies a similar stretched exponential decay when $u < \tilde{u}$, and importantly that

$$\tilde{u} > 0, \quad \text{when } \nu \geq 1. \quad (0.16)$$

When $E = \mathbb{Z}^{d+1}$, $d \geq 2$, for all the above results, the distance in (0.3) and the associated balls can be replaced with the more common sup-norm distance and corresponding balls. Even in the special case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, the above results improve on present knowledge, see Remarks 4.2 (1) and 5.6 (1).

However in the case of a general E with $\nu < 1$ (for instance when G is the discrete skeleton of the Sierpinski gasket), it is a challenging question left open by the present work to understand whether $u_* > 0$, see Remark 5.6 (2).

We now provide some comments on the proofs. An important difficulty stems from the long range interaction present in random interlacements, (for instance the correlation of the events $\{x \in \mathcal{V}^u\}$ and $\{x' \in \mathcal{V}^u\}$ decays as $d(x, x')^{-\nu}$, when the distance between x and x' grows, see (1.38)). One important novelty of the present work, (even in the classical case $E = \mathbb{Z}^{d+1}$, $d \geq 2$), is that we use the same renormalization scheme to treat both the percolative and non-percolative regimes of \mathcal{V}^u . The heart of the matter is encapsulated in the *decoupling inequalities* stated in Theorem 2.6.

The decoupling inequalities bound from above the probability of an intersection of 2^n decreasing events, or 2^n increasing events that depend on the respective traces of random interlacements in 2^n boxes in E , which are well “spread out”, and can be thought of as the “bottom leaves” of a dyadic tree of depth n .

More precisely, for 2^n decreasing, resp. increasing, events A_m , resp. B_m , on $\{0, 1\}^E$, labelled by m , respectively adapted to 2^n balls of size of order L_0 , which are well “spread out”, this last feature involves a geometrically growing sequence of length scales $L_n = \ell_0^n L_0$, see (2.1)–(2.5), the decoupling inequalities state that for $K > 0$, $0 < \nu' < \nu = \alpha - \frac{\beta}{2}$, when $\ell_0 \geq c(K, \nu')$, $L_0 \geq 1$, $u > 0$, one has

$$\mathbb{P}\left[\bigcap_m A_m^{u+}\right] \leq \prod_m (\mathbb{P}[A_m^u] + \varepsilon(u)), \quad \text{and} \quad (0.17)$$

$$\mathbb{P}\left[\bigcap_m B_m^{u-}\right] \leq \prod_m (\mathbb{P}[B_m^u] + \varepsilon(u_-)), \quad (0.18)$$

where

$$u_{\pm} = \prod_{n \geq 0} \left(1 + c_1 \sqrt{\frac{K}{(n+1)^3}} \ell_0^{-\frac{(\nu-\nu')}{2}}\right)^{\pm 1} u, \quad (0.19)$$

and we have set for $\nu > 0$,

$$\varepsilon(\nu) = 2e^{-K\nu L_0^{\nu} \ell_0^{\nu'}} / (1 - e^{-K\nu L_0^{\nu} \ell_0^{\nu'}}). \quad (0.20)$$

The notation A_m^{ν} , resp. B_m^{ν} , in (0.17), resp. (0.18), refers to the event where the indicator function of \mathcal{I}^{ν} belongs to A_m , resp. B_m .

The tuning of the level u into u_+ or u_- in (0.17), (0.18), corresponds to the “sprinkling technique”, where throwing in some additional trajectories of the random interlacements provides a way to dominate long range interactions. Several variations of this strategy have already been employed, see

[18, 19, 21]. The novelty here is that percolative and non-percolative regimes are handled in a unified and more powerful fashion thanks to the decoupling inequalities, which themselves are the consequence of the renormalization step stated in Theorem 2.1.

The strength of the decoupling inequalities comes to bear when applied to *cascading families* of events as discussed in Sect. 3. Informally, the occurrence of an event of such a family on a “large scale” trickles down to a “lower scale”, and permits the construction of dyadic trees of finite depth, forcing the occurrence of events of the family at the bottom scale, in well spread out boxes corresponding to the leaves of the tree. Balancing out the combinatorial complexity of the possible dyadic trees arising in this construction, with the bounds coming from the decoupling inequalities (0.17), (0.18), is the key to the derivation of the stretched exponential bounds in (0.12), (0.15). The approach we develop here also gives a way to revisit [27] from a new perspective, see Remarks 3.3 (2) and 3.8 (1).

We will now explain the structure of this article.

In Sect. 1 we introduce further notation and collect several useful results about the weighted graph E , random walk on E , and random interacements on E . The Harnack inequality that appears in Lemma 1.2 is instrumental in Sect. 2.

Section 2 develops the renormalization scheme underpinning the decoupling inequalities of Theorem 2.6. The key renormalization step is carried out in Theorem 2.1.

In Sect. 3 we bring into play the cascading property and give several examples in Proposition 3.2 and Remark 3.3. The combination of this notion with the decoupling inequalities leads to the bounds stated in Theorem 3.4. Some consequences are stated in the Corollaries 3.5 and 3.7.

Section 4 applies Theorem 3.4 and Corollary 3.5 to prove the finiteness of u_{**} , (and hence of u_*), as well as the stretched exponential decay of the connectivity function when $u > u_{**}$, see (0.10)–(0.12). The main result appears in Theorem 4.1.

In Sect. 5 we apply Corollary 3.5 and 3.7 to show in Theorem 5.1 that $\tilde{u} \leq u_*$, derive for $u < \tilde{u}$ a stretched exponential bound for the occurrence of a large finite cluster in the intersection of \mathcal{V}^u with a half-space, and establish that $\tilde{u} > 0$, when $v \geq 1$.

The Appendix provides the proof of the cascading property of the family of “separation events” introduced in Remark 3.3 (2), in the spirit of [27].

Let us finally explain the convention we use concerning constants. We denote with $c, c', \tilde{c}, \bar{c}$ positive constants changing from place to place, which only depend on the weighted graph G (and in particular on α and β). Numbered constants c_0, c_1, \dots refer to the value corresponding to their first appearance in the text. Finally dependence of constants on additional param-

eters appears in the notation. For instance $c(u)$ denotes a positive constant depending on the weighted graph G and on u .

1 Notation and some useful facts

In this section we introduce the precise set-up and collect various useful results about random walks and random interlacements on the type of weighted graphs we consider in this article. An important control on Harnack constants for harmonic functions in $d(\cdot, \cdot)$ -balls is stated in Lemma 1.2. Further we collect some facts concerning capacity and entrance probabilities in Lemma 1.3. Several useful features of random interlacements appear in Lemma 1.4.

We let $\mathbb{N} = \{0, 1, \dots\}$ denote the set of natural numbers. When u is a non-negative real number, we let $[u]$ stand for the integer part of u . Given a finite set A we denote with $|A|$ its cardinality. The graphs we consider have a countable vertex set and an edge set made of unordered pairs of the vertex set. With an abuse of notation we often denote a graph by its vertex set, when this causes no confusion. When x, x' are distinct vertices of the graph, we write $x \sim x'$, if x and x' are neighbors, i.e. if $\{x, x'\}$ is an edge of the graph. The graphs we discuss here are connected and have bounded degree, i.e. each vertex has a uniformly bounded number of neighbors. A finite path in a graph refers to a sequence x_0, \dots, x_N , $N \geq 0$, of vertices of the graph such that $x_i \sim x_{i+1}$, for $0 \leq i < N$. We sometimes write path instead of finite path, when this causes no confusion. Given a graph Γ as above, we denote with $d_\Gamma(\cdot, \cdot)$ the graph-distance on Γ , i.e. the minimal number of steps of a finite path joining two given vertices of Γ . The graphs under consideration being connected, this number is automatically finite. We write $B_\Gamma(x, r)$ for the closed d_Γ -ball with center x in Γ and radius $r \geq 0$. When U is a subset of vertices in Γ , we denote by ∂U , $\partial_{\text{int}} U$, and \overline{U} , the vertex boundary, the interior vertex boundary, and the closure of U :

$$\begin{aligned} \partial U &= \{x \in U^c; \exists x' \in U, \text{ with } x \sim x'\}, \\ \partial_{\text{int}} U &= \{x \in U; \exists x' \in U^c, \text{ with } x \sim x'\}, \quad \text{and} \quad \overline{U} = U \cup \partial U. \end{aligned} \quad (1.1)$$

Further we denote by χ_U the indicator function of U , and write $U \subset\subset \Gamma$ to express that U is a finite subset of the vertex set of Γ . A weight on Γ is a symmetric non-negative function $\rho_{x,x'}$ on $\Gamma \times \Gamma$ such that $\rho_{x,x'} > 0$ if and only if $x \sim x'$. A weight ρ induces a measure on the vertex set of Γ via

$$\rho(x) = \sum_{x' \sim x} \rho_{x,x'} \quad \text{and} \quad \rho(U) = \sum_{x \in U} \rho(x), \quad \text{for } U \subseteq \Gamma. \quad (1.2)$$

The natural weight on Γ refers to the choice $\rho_{x,x'} = 1_{x \sim x'}$, and in this case $\rho(x)$ coincides with the degree of the vertex x in Γ . The set \mathbb{Z}^m , $m \geq 1$,

throughout this work is tacitly endowed with its usual graph structure and its natural weight, unless explicitly stated otherwise.

A weighted graph (Γ, ρ) induces a random walk on the vertex set of Γ having transition probability

$$p_{x,x'} = \rho_{x,x'}/\rho(x), \quad \text{for } x, x' \text{ in } \Gamma. \quad (1.3)$$

It satisfies the detailed balance equations relative to ρ :

$$\rho(x) p_{x,x'} = \rho(x') p_{x'x}, \quad \text{for } x, x' \text{ in } \Gamma. \quad (1.4)$$

We let P_x stand for the canonical law of the walk on Γ starting at x , with transition probability as in (1.3), and denote by $(X_n)_{n \geq 0}$ the canonical process. The transition density of the walk is defined as follows:

$$p_n(x, x') = P_x[X_n = x']/\rho(x'), \quad \text{for } n \geq 0, x, x' \text{ in } \Gamma. \quad (1.5)$$

It is a symmetric function of x, x' thanks to (1.4).

As explained in the Introduction our main interest lies in graphs of the form $E = G \times \mathbb{Z}$, where G is an infinite connected graph of bounded degree endowed with a weight ρ^G such that $\rho_{x,x'}^G$ is uniformly bounded, and uniformly positive, when $x \sim x'$. We then endow E with the weight:

$$\begin{aligned} \rho_{x,x'} &= \rho_{y,y'}^G, & \text{if } z = z', & \quad \text{with } x = (y, z), x' = (y', z') \text{ in } E, \\ &= 1, & \text{if } |z - z'| = 1 \text{ and } y = y', & \\ &= 0, & \text{otherwise.} & \end{aligned} \quad (1.6)$$

In particular for x as above, $\rho(x) = \rho^G(y) + 2$, see (1.2).

We further assume that for some $\alpha > 1$, (G, ρ^G) is α -Ahlfors regular, that is

$$c R^\alpha \leq \rho^G(B_G(y, R)) \leq c' R^\alpha, \quad \text{for all } R \geq \frac{1}{2} \text{ and } y \text{ in } G. \quad (1.7)$$

When referring to the random walk in the weighted graph (E, ρ) , we use the notation introduced below (1.4). The Green density is defined as follows:

$$g(x, x') = \sum_{n \geq 0} p_n(x, x'), \quad \text{for } x, x' \text{ in } E. \quad (1.8)$$

It is a symmetric function. It is in fact finite. To see this point, one uses (1.7) combined with the upper bound from Proposition 3.13 of [17] on the transition density of the continuous time walk on G using the weights on G as jump rates. The transition density of the analogous continuous time walk on E is

then the product of the transition densities of the continuous time walk on G and of the continuous time walk on \mathbb{Z} . Relating $g(\cdot, \cdot)$ to the Green density of the continuous time walk on E , the finiteness of $g(\cdot, \cdot)$ readily follows. We also refer to the Appendix of [20] for related controls. In a more substantive fashion, we crucially assume the existence of a number β in $[2, 1 + \alpha]$ such that

$$c(d(x, x') \vee 1)^{-\nu} \leq g(x, x') \leq c'(d(x, x') \vee 1)^{-\nu}, \quad \text{for } x, x' \text{ in } E, \quad (1.9)$$

where $\nu = \alpha - \frac{\beta}{2}$ (> 0), and $d(\cdot, \cdot)$ denotes the distance function on E as in (0.3), that is:

$$d(x, x') = \max\{d_G(y, y'), |z - z'|^{\frac{2}{\beta}}\}, \quad \text{for } x = (y, z), x' = (y', z') \text{ in } E. \quad (1.10)$$

Remark 1.1

- (1) The condition $\beta \geq 2$ is natural since we want that $d(\cdot, \cdot)$ in (1.10) defines a metric. The inequality $\beta \leq 1 + \alpha$ is then a consequence of (1.7)–(1.10). Indeed by (1.9), (1.11), bringing into play the killed Green densities (see also for instance (1.18) below), one sees that $E_x[T_{B(x, R)}] \geq cR^\beta$, see (1.12) for notation, and a similar inequality then holds for the walk on G , with $B_G(y, R)$ in place of $B(x, R)$. The inequality $\beta \leq 1 + \alpha$ now follows from a similar argument as in Lemma 1.2 of [2].
- (2) Under the assumptions on (G, ρ^G) stated above (1.6), for given $\alpha > 1$, $2 \leq \beta \leq \alpha + 1$, the conditions (1.7) and (1.9) are actually equivalent to the sub-Gaussian bounds (0.4) on the probability density $p_n^G(y, y')$ of the walk on G , as we now explain.

Under (0.4), condition (1.7) follows from Grigoryan-Telcs [11], p. 503–504. The proof of Lemma 5.4 of [20], with obvious modifications since ρ^G is the natural weight in [20], shows that (1.9) holds.

Conversely given (1.7) and (1.9), the controls on Harnack constants for positive harmonic functions in $d(\cdot, \cdot)$ -balls from Lemma 1.2 below, imply, in the terminology of [12], the elliptic Harnack inequality on (G, ρ^G) . In addition, for the walk on G , when starting at y , the expected exit time from $B_G(y, R)$ is at least cR^β , as mentioned in 1) above. It is also at most $c'R^\beta$. To see this last point, one can for instance use the following fact, which is a consequence of (1.9) and of lower bounds on the killed Green densities as in (1.20) below, that given x and x' in E with $d(x, x') \leq 3R$, the walk on E enters $B(x', R)$ before exiting $B(x, cR)$, with a probability uniformly bounded away from 0, when c is chosen large. Condition (0.4) now follows from Theorem 3.1 of Grigoryan-Telcs [12].

We refer to [2, 3, 11, 12, 14], for further examples and equivalent characterizations of (0.4) combined with the assumptions stated above (1.6).

The metric $d(\cdot, \cdot)$ in (1.10) is well adapted to the possibly different natures of the horizontal and vertical displacements of the walk on E . It plays an important role throughout this work. We write $B(x, r)$ for the corresponding closed ball with center $x \in E$ and radius $r \geq 0$. When K, K' are subsets of E we write $d(K, K') = \inf\{d(x, x'); x \in K, x' \in K'\}$ for the mutual distance of K and K' . When $K = \{x\}$ is a singleton, we simply write $d(x, K')$. As a direct consequence of (1.6), (1.7), (1.10) we have the following control on the volume of balls in E :

$$c R^{\alpha + \frac{\beta}{2}} \leq \rho(B(x, R)) \leq c' R^{\alpha + \frac{\beta}{2}}, \quad \text{for } x \in E, R \geq \frac{1}{2}. \quad (1.11)$$

In the important special case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, (i.e. when $G = \mathbb{Z}^d$, $d \geq 2$), we denote with $d_\infty(\cdot, \cdot)$ the sup-norm distance on \mathbb{Z}^{d+1} and write $B_\infty(x, r)$ for the corresponding closed ball with center x and radius r . Let us already point out that when $E = \mathbb{Z}^{d+1}$ we can replace $d(\cdot, \cdot)$ with the more common metric $d_\infty(\cdot, \cdot)$ in the various constructions and results we develop in this work.

We denote by W_+ the set of nearest neighbor trajectories on E , defined for non-negative times and tending to infinity, and by \mathcal{W}_+ the canonical σ -algebra on W_+ induced by the canonical process X_n , $n \geq 0$. Due to (1.9), the walk on (E, ρ) is transient and the set W_+ has full measure under any P_x . From now on we will view P_x as a measure on (W_+, \mathcal{W}_+) . When m is a measure on E , we denote by P_m the measure $\sum_{x \in E} m(x) P_x$, and by E_m the corresponding expectation. We let θ_n , $n \geq 0$, stand for the canonical shift on W_+ , so that $\theta_n(w)(\cdot) = w(n + \cdot)$, for $w \in W_+$, $n \geq 0$. Given $U \subseteq E$, we write

$$\begin{aligned} H_U &= \inf\{n \geq 0; X_n \in U\}, & \tilde{H}_U &= \inf\{n \geq 1; X_n \in U\} \\ T_U &= \inf\{n \geq 0; X_n \notin U\}, \end{aligned} \quad (1.12)$$

for the entrance time in U , the hitting time of U , and the exit time from U . In the case $U = \{x\}$, we write H_x and \tilde{H}_x for simplicity.

We now come to some important controls on Harnack constants. Given $U \subseteq E$, we say that a function on \overline{U} , see (1.1) for notation, is harmonic in U , when in the notation of (1.3)

$$\sum_{x' \sim x} p_{x, x'} f(x') = f(x), \quad \text{for all } x \text{ in } U. \quad (1.13)$$

The following lemma is a crucial ingredient for the implementation of the sprinkling technique in Sect. 2. It is based on (1.9) and Lemma A.2 of [25], which is an adaptation of Lemma of 10.2 of [11].

Lemma 1.2 *There exist $c, c_0 > 1$ such that for $x \in E$, $L \geq 1$, and v a non-negative function on $\overline{B}(x, c_0 L)$, harmonic in $B(x, c_0 L)$ one has*

$$\max_{B(x, L)} v \leq c \min_{B(x, L)} v. \quad (1.14)$$

Proof We define $U_i = B(x, L_i)$, for $i = 1, 2, 3$, where $L_1 = L$, $L_2 = 3L$, and $L_3 \geq L_2$ so that $U_1 \subseteq U_2 \subseteq U_3$. It follows from Lemma A.2 of [25] that when v' is a non-negative function on \overline{U}_3 harmonic in U_3 , one has

$$\max_{U_1} v \leq K \min_{U_1} v, \quad (1.15)$$

where

$$K = \max_{\tilde{x}, \bar{x} \in U_1} \max_{x' \in \partial_{\text{int}} U_2} g_{U_3}(\tilde{x}, x') / g_{U_3}(\bar{x}, x'), \quad (1.16)$$

and $g_{U_3}(\cdot, \cdot)$ denotes the killed Green density:

$$g_{U_3}(\tilde{x}, x') = E_{\tilde{x}} \left[\sum_{k \geq 0} 1\{X_k = x', T_{U_3} > k\} \right] / \rho(x'), \quad \text{for } \tilde{x}, x' \text{ in } E. \quad (1.17)$$

Applying the strong Markov property at time T_{U_3} we find that

$$g_{U_3}(\tilde{x}, x') = g(\tilde{x}, x') - E_{\tilde{x}}[g(X_{T_{U_3}}, x')], \quad \text{for } \tilde{x}, x' \text{ in } E. \quad (1.18)$$

As a result when \tilde{x}, \bar{x} belong to U_1 and x' to $\partial_{\text{int}} U_2$, it follows from (1.9) that

$$g_{U_3}(\tilde{x}, x') \leq g(\tilde{x}, x') \leq c(L_2 - L_1 - 1)^{-\nu} \leq c L^{-\nu}, \quad (1.19)$$

$$g_{U_3}(\bar{x}, x') \geq c L_2^{-\nu} - c(L_3 - L_2)^{-\nu} \geq c L^{-\nu}, \quad \text{when } L_3 = c_0 L. \quad (1.20)$$

The claim (1.14) now follows from (1.15), (1.16), together with $B(x, c_0 L) = U_3$ and $B(x, L) = U_1$. \square

We now recall some facts concerning the equilibrium measure and the capacity of a finite subset of E . Given $K \subset \subset E$, (see below (1.1) for the notation) we write e_K for the equilibrium measure of K and $\text{cap}(K)$ for its total mass, the capacity of K :

$$\begin{aligned} e_K(x) &= P_x[\tilde{H}_K = \infty] \rho(x) 1_K(x), \quad \text{for } x \in E, \\ \text{cap}(K) &= e_K(E) = \sum_{x \in K} P_x[\tilde{H}_K = \infty] \rho(x). \end{aligned} \quad (1.21)$$

The subadditive property of the capacity easily follows from (1.21):

$$\text{cap}(K \cup K') \leq \text{cap}(K) + \text{cap}(K'), \quad \text{for } K, K' \subset \subset E, \quad (1.22)$$

and classically, one can express the probability that the walk enters K via:

$$P_x[H_K < \infty] = \sum_{x' \in K} g(x, x') e_K(x'), \quad \text{for } x \in E. \quad (1.23)$$

The following lemma collects further useful facts.

Lemma 1.3

$$\begin{aligned} \text{cap}(\{x\}) &= g(x, x)^{-1}, \\ \text{cap}(\{x, x'\}) &= \frac{g(x, x) + g(x', x') - 2g(x, x')}{g(x, x)g(x', x') - g(x, x')^2}, \quad \text{for } x \neq x' \text{ in } E. \end{aligned} \quad (1.24)$$

For $K \subset \subset E$, $x \in E$, one has:

$$\begin{aligned} \sum_{x' \in K} g(x, x') / \sup_{\bar{x} \in K} \sum_{x' \in K} g(\bar{x}, x') &\leq P_x[H_K < \infty] \\ &\leq \sum_{x' \in K} g(x, x') / \inf_{\bar{x} \in K} \sum_{x' \in K} g(\bar{x}, x'), \end{aligned} \quad (1.25)$$

$$c L^\nu \leq \text{cap}(B(x, L)) \leq c' L^\nu, \quad \text{for } x \in E, L \geq \frac{1}{2}. \quad (1.26)$$

Proof The claim (1.24) follows by writing $e_{\{x\}} = \gamma_x \delta_x$ and $e_{\{x, x'\}} = \lambda_x \delta_x + \lambda_{x'} \delta_{x'}$, with $\gamma_x, \lambda_x, \lambda_{x'} \geq 0$, and then using (1.23) together with the identity $P_{\bar{x}}[H_K < \infty] = 1$, when $\bar{x} \in K$, to determine $\gamma_x, \lambda_x, \lambda_{x'}$.

The bound (1.25) follows in a classical fashion from the $L^1(P_x)$ -convergence of the bounded martingale $M_n = \sum_{x' \in K} g(X_{m \wedge H_K}, x')$, $n \geq 0$, towards $1\{H_K < \infty\} \sum_{x' \in K} g(X_{H_K}, x')$, and the resulting equality of the P_x -expectation of this quantity with $\sum_{x' \in K} g(x, x')$.

As for (1.26), a variation of the above argument yields the identity

$$g(x, x') = E_x[g(X_{H_B}, x'), H_B < \infty], \quad \text{for } x, x' \text{ in } E \text{ and } B = B(x', L).$$

By (1.9) we thus see that for $x \notin B$ one has

$$c(L/d(x, x'))^\nu \leq P_x[H_B < \infty] \stackrel{(1.23)}{=} \sum_{\bar{x} \in B} g(x, \bar{x}) e_B(\bar{x}) \leq c'(L/d(x, x'))^\nu.$$

Letting x tend to infinity the claim (1.26) now follows from (1.9). \square

We now collect some properties of random interacements on the transient weighted graph (E, ρ) , which will be useful in the following sections. Random interacements on E are defined on a canonical space $(\Omega, \mathcal{A}, \mathbb{P})$. On this

space a certain Poisson point process on $W^* \times \mathbb{R}_+$, (with W^* the space of doubly infinite E -valued trajectories tending to infinity at positive and negative infinite times, modulo time-shift), having intensity measure $\nu(dw^*)du$, can be constructed. We refer to Sect. 2 of Teixeira [26], and also to Remark 1.4 of [21], for the precise definition of this probability space and of the σ -finite measure $\nu(dw^*)$. For the purpose of the present work we only need to recall that one can define on $(\Omega, \mathcal{A}, \mathbb{P})$ some families of Poisson point processes on the space W_+ , see above (1.12) for the notation, namely $\mu_{K,u}(dw)$ and $\mu_{K,u',u}(dw)$, for $K \subset \subset E$, and $u, u' \geq 0$, with $u' < u$, so that

$$\begin{aligned} \mu_{K,u',u} \text{ and } \mu_{K,u'} \text{ are independent with respective intensity measures} \\ (u - u')P_{e_K} \text{ and } u'P_{e_K}, \end{aligned} \quad (1.27)$$

and

$$\mu_{K,u} = \mu_{K,u'} + \mu_{K,u',u}. \quad (1.28)$$

In essence $\mu_{K,u}$, resp. $\mu_{K,u',u}$, keep track of the part after entrance in K of the doubly infinite trajectories modulo time-shift, with labels at most u , resp. with labels in $(u', u]$, belonging to the canonical Poisson cloud, that do enter K . These finite point measures on W_+ satisfy certain compatibility relations, the *sweeping identities*:

$$\begin{aligned} \mu_{K,u} &= \sum_{i=1}^m 1\{H_K(w_i) < \infty\} \delta_{\theta_{H_K(w_i)}}, \\ \text{if } \mu_{K',u} &= \sum_{i=1}^m \delta_{w_i}, \quad \text{for } K \subset K' \subset \subset E, u \geq 0, \end{aligned} \quad (1.29)$$

together with the analogous relations for $\mu_{K,u',u}$ and $\mu_{K',u',u}$, $0 \leq u' < u$. We refer to (1.13)–(1.15) of [19] or to (1.18)–(1.21) and Proposition 1.3 of [21] for more details.

Given $\omega \in \Omega$, the random interlacement at level $u \geq 0$ is the random subset of E defined by

$$\mathcal{I}^u(\omega) = \bigcup_{K \subset \subset E} \bigcup_{w \in \text{Supp } \mu_{K,u}(\omega)} \text{range}(w), \quad (1.30)$$

where the notation $\text{Supp } \mu_{K,u}(\omega)$ refers to the support of the finite point measure $\mu_{K,u}(dw)$ and $\text{range}(w) = w(\mathbb{N})$ for $w \in W_+$. The vacant set at level u is then defined as

$$\mathcal{V}^u(\omega) = E \setminus \mathcal{I}^u(\omega), \quad \text{for } \omega \in \Omega, u \geq 0. \quad (1.31)$$

As a straightforward consequence of the compatibility relations (1.29), see for instance (1.54) of [21], one finds that:

$$\begin{aligned} \mathcal{I}^u(\omega) \cap K &= \bigcup_{w \in \text{Supp } \mu_{K',u}(\omega)} w(\mathbb{N}) \cap K, \\ \text{for any } K \subset K' \subset \subset E, u \geq 0, \omega \in \Omega. \end{aligned} \quad (1.32)$$

It also readily follows from (1.27) that

$$\mathbb{P}[\mathcal{V}^u \supseteq K] = \exp\{-u \text{cap}(K)\}, \quad \text{for all } K \subset \subset E, u \geq 0. \quad (1.33)$$

This identity determines the law Q_u on $\{0, 1\}^E$ of $\chi_{\mathcal{V}^u}$, the indicator function of \mathcal{V}^u , see Remark 2.2 (2) of [21], or Remark 2.3 of [26]. The law Q_u satisfies the FKG Inequality, see Theorem 3.1 of [26]: when f, g are increasing, $\sigma(\Psi_x, x \in E)$ -measurable functions on $\{0, 1\}^E$, (with $\Psi_x, x \in E$, the canonical coordinates on $\{0, 1\}^E$), which are Q_u -square integrable, then

$$E^{Q_u}[fg] \geq E^{Q_u}[f] E^{Q_u}[g]. \quad (1.34)$$

The following notation will be quite handy and recurrently used in the sequel. Given $\mathcal{J}(\omega)$ a random subset of E , and A a subset of $\{0, 1\}^E$, we define

$$A(\mathcal{J}) = \{\omega \in \Omega; \chi_{\mathcal{J}(\omega)} \in A\}, \quad (1.35)$$

and write for $u \geq 0$:

$$A^u \stackrel{\text{def}}{=} A(\mathcal{I}^u). \quad (1.36)$$

The following lemma attests the presence of a long range dependence in random interacements, and states a zero-one law for the occurrence of an infinite cluster in \mathcal{V}^u . We recall that $\nu = \alpha - \frac{\beta}{2} > 0$, see (1.9).

Lemma 1.4 ($u > 0, x, x'$ in E)

$$\mathbb{P}[x \in \mathcal{V}^u] = \exp\left\{-\frac{u}{g(x, x)}\right\}, \quad (1.37)$$

$$\begin{aligned} c(u)d(x, x')^{-\nu} &\leq \text{cov}_{\mathbb{P}}(1\{x \in \mathcal{V}^u\}, 1\{x' \in \mathcal{V}^u\}) \leq c'(u)d(x, x')^{-\nu}, \\ \text{for } x \neq x'. \end{aligned} \quad (1.38)$$

$$\mathbb{P}[\mathcal{V}^u \text{ contains an infinite connected component}] \in \{0, 1\}. \quad (1.39)$$

Proof The identity (1.37) directly follows from (1.24), (1.33). Similarly the covariance in (1.38) is equal to, (writing g_x in place of $g(x, x)$):

$$\exp\{-u(g_x^{-1} + g_{x'}^{-1})\} \left(\exp\left\{u \left[\frac{g_x + g_{x'}}{g_x g_{x'}} - \frac{g_x + g_{x'} - 2g(x, x')}{g_x g_{x'} - g(x, x')^2} \right] \right\} - 1 \right). \quad (1.40)$$

The expression inside the square bracket is non-negative as a result of (1.24) and the subadditivity property of the capacity, see (1.22). The claim (1.38) when $d(x, x') > c(u)$ is a simple consequence of (1.9). The case $0 < d(x, x') \leq c(u)$ follows from the inequality

$$g(x, x')/g_{x'} \leq \tilde{c} < 1 \quad (\text{and similarly for } g(x', x)/g_x), \quad (1.41)$$

which straightforwardly implies that the expression in the square bracket remains uniformly bounded away from zero for such x, x' . To prove (1.41) one simply notes that $P_x[H_{x'} < \infty] = g(x, x')/g_{x'}$, and uses an escape route from x' for the walk starting at x , which first moves in the \mathbb{Z} -direction and then takes advantage of (1.9) to show that the above probability is smaller than $\tilde{c} < 1$.

Let us finally prove (1.39). When $z \in \mathbb{Z}$, the translation t_z in the \mathbb{Z} -direction on $\{0, 1\}^E$ is defined for $\sigma \in \{0, 1\}^E$ via $t_z(\sigma) = \sigma(x + z)$, for $x \in E$, where $x + z$ is the natural translation of x by z in $E = G \times \mathbb{Z}$. As a direct consequence of (1.33) and the fact that $\text{cap}(K + z) = \text{cap}(K)$ for any $K \subset \subset E$, we see that $t_z, z \in \mathbb{Z}$, preserves Q_u . Similar arguments as in the proof of Theorem 2.1 of [21], see in particular (2.6), (2.15) of [21], show that the group $t_z, z \in \mathbb{Z}$ of Q_u -preserving transformations is ergodic, (see also (1.42) below). The claim (1.39) readily follows. \square

Remark 1.5

- (1) The arguments used in the proof of (2.15) of [21] show that when K, K' are disjoint finite subsets of E , and F, F' are bounded measurable functions on the set of finite point measures on W_+ , then for $u \geq 0$, one has

$$|\text{cov}_{\mathbb{P}}(F(\mu_{K,u}), F'(\mu_{K',u}))| \leq c u \frac{\text{cap}(K) \text{cap}(K')}{d(K, K')^v} \|F\|_{\infty} \|F'\|_{\infty}, \quad (1.42)$$

with $d(K, K')$ as above (1.11), and $\|F\|_{\infty}, \|F'\|_{\infty}$ the respective sup-norms of F and F' .

- (2) In the case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, the definition of the capacity in (1.21) differs by a multiplicative factor $2(d+1)$ from the definition in [21], due to the presence of $\rho(\cdot)$. The convention used in [21] correspond to assigning a weight $(2d+2)^{-1}$ to each edge of \mathbb{Z}^{d+1} , so that the volume measure coincides with the counting measure. This multiplicative factor leads to the

- (in most respect unessential) fact that random interlacement at level u in the present work correspond to random interlacements at level $2(d+1)u$ in the terminology of [18, 19, 21].
- (3) In the context of Bernoulli percolation two inequalities play an important role in the development of the theory, the BK Inequality and the FKG Inequality, see for instance [13]. We have seen in (1.34) that the FKG Inequality holds for Q_u . However it follows from (1.38) that for $x \neq x'$ the events $\{x \in \mathcal{V}^u\}$ and $\{x' \in \mathcal{V}^u\}$ are correlated, in fact positively correlated, when $u > 0$. As a result the BK Inequality does not hold for Q_u . The decoupling inequalities of Theorem 2.6 in the following section play the role of a partial substitute for the absence of a BK Inequality.

2 Decoupling inequalities and the sprinkling technique

In this section we implement a certain renormalization scheme. The goal is to derive decoupling inequalities for the probability of intersections of 2^n events, which are either all increasing or all decreasing, and respectively pertain to the state of random interlacements on E at a certain level, in 2^n boxes of typical length L_0 . These boxes should be thought of as the bottom leaves of a dyadic tree of depth n , and a geometrically increasing sequence of length scales, $L_n = \ell_0^n L_0$, $n \geq 0$, is used to quantify the fact that these boxes are well spread out. The key idea to overcome the presence of long range interactions in the model is the sprinkling technique. Roughly speaking one slightly increases the level so as to throw in more trajectories and dominate terms containing interactions between distant boxes at a given scale. The fashion in which we conduct the sprinkling technique is new, even in the case of $E = \mathbb{Z}^{d+1}$, $d \geq 2$, and differs markedly from [18, 19, 21]. The key induction step appears in Theorem 2.1 and the crucial decoupling inequalities are presented in Theorem 2.6. In the important special case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, one can replace the distance $d(\cdot, \cdot)$ with the sup-norm distance $d_\infty(\cdot, \cdot)$ and the balls $B(x, r)$ with the corresponding balls $B_\infty(x, r)$ throughout the article, simply adapting constants when necessary.

We introduce a geometrically growing sequence of length scales (we recall that $c_0 > 1$ appears in Lemma 1.2):

$$L_n = \ell_0^n L_0, \quad \text{for } n \geq 0, \text{ where } L_0 \geq 1 \text{ and } \ell_0 \geq 10^6 c_0. \quad (2.1)$$

We want to derive upper bounds on the probability of intersections of events, which occur in boxes of size L_0 , which are well spread-out. Specifically this last feature will bring into play certain embeddings of dyadic trees of finite depth n , which we now describe.

Given $n \geq 0$, we denote with $T_n = \bigcup_{0 \leq k \leq n} \{1, 2\}^k$, the canonical dyadic tree of depth n and with $T_{(k)} = \{1, 2\}^k$, the collection of elements of the tree

at depth k . We write Λ_n for the set of *embeddings* of T_n in E , that is of maps $\mathcal{T} : T_n \rightarrow E$, such that defining

$$x_{m,\mathcal{T}} = \mathcal{T}(m), \quad \tilde{C}_{m,\mathcal{T}} = B(x_{m,\mathcal{T}}, 10L_{n-k}), \quad \text{for } m \in T_{(k)}, \quad 0 \leq k \leq n, \quad (2.2)$$

one has for any $0 \leq k < n$, $m \in T_{(k)}$, and m_1, m_2 the two descendants of m in $T_{(k+1)}$ obtained by respectively concatenating 1 and 2 to m :

$$\tilde{C}_{m_1,\mathcal{T}} \cup \tilde{C}_{m_2,\mathcal{T}} \subseteq \tilde{C}_{m,\mathcal{T}}, \quad (2.3)$$

$$d(x_{m_1,\mathcal{T}}, x_{m_2,\mathcal{T}}) \geq \frac{1}{100} L_{n-k}. \quad (2.4)$$

Due to (2.1) it follows that for each $0 \leq k \leq n$, the “boxes at depth k ”, $\tilde{C}_{m,\mathcal{T}}$, $m \in T_{(k)}$, are pairwise disjoint.

Given $n \geq 0$ and $\mathcal{T} \in \Lambda_n$, a collection of measurable subsets of $\{0, 1\}^E$ indexed by the “leaves” of T_n , A_m , $m \in T_{(n)}$, is said \mathcal{T} -*adapted* when (see above (1.34) for notation):

$$A_m \text{ is } \sigma(\Psi_x, x \in \tilde{C}_{m,\mathcal{T}})\text{-measurable for each } m \in T_{(n)}. \quad (2.5)$$

We also recall that given $u \geq 0$, the events $A_m^u (\subseteq \Omega)$, $m \in T_{(n)}$, are defined in (1.36).

We still need some notation before stating the main induction step of the renormalization scheme we develop in this section. Given $n \geq 0$ and $\mathcal{T} \in \Lambda_{n+1}$, we denote with $\mathcal{T}_1, \mathcal{T}_2 \in \Lambda_n$, the two embeddings of T_n such that $\mathcal{T}_i(m) = \mathcal{T}((i, i_1, \dots, i_k))$ for $i = 1, 2$ and $m = (i_1, \dots, i_k)$ in $T_{(k)}$, $0 \leq k \leq n$. Loosely speaking \mathcal{T}_1 and \mathcal{T}_2 are the two natural embeddings corresponding to the respective restrictions of \mathcal{T} to the descendants of 1 and of 2 in T_{n+1} . Given a \mathcal{T} -adapted collection A_m , $m \in T_{(n+1)}$, we then define two collections $A_{m,1}$, $m \in T_{(n)}$, and $A_{m,2}$, $m \in T_{(n)}$, respectively \mathcal{T}_1 - and \mathcal{T}_2 -adapted, as follows:

$$A_{m,i} = A_{(i,i_1,\dots,i_n)}, \quad \text{for } i = 1, 2 \text{ and } m = (i_1, \dots, i_n) \in T_{(n)}. \quad (2.6)$$

We can now state the main induction step.

Theorem 2.1 ($K > 0$, $0 < v' < v \stackrel{(1.9)}{=} \alpha - \frac{\beta}{2}$)

When $\ell_0 \geq c(K, v')$, then for all $n \geq 0$, $\mathcal{T} \in \Lambda_{n+1}$, for all \mathcal{T} -adapted collections A_m , $m \in T_{(n+1)}$, of decreasing events, respectively B_m , $m \in T_{(n+1)}$, of increasing events, on $\{0, 1\}^E$, and for all $0 < u' < u$ such that

$$u \geq (1 + c_1 \sqrt{K} (n+1)^{-\frac{3}{2}} \ell_0^{-\frac{(v-v')}{2}}) u', \quad (2.7)$$

one has

$$\mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} A_m^u\right] \leq \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} A_{\bar{m}_1,1}^u\right] \left(\mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} A_{\bar{m}_2,2}^{u'}\right] + 2e^{-2u' \frac{K}{(n+1)^3} L_n^v \ell_0^{v'}}\right), \quad (2.8)$$

respectively,

$$\begin{aligned} \mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} B_m^{u'}\right] &\leq \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} B_{\bar{m}_1,1}^u\right] \mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} B_{\bar{m}_2,2}^u\right] \\ &\quad + 2e^{-2u' \frac{K}{(n+1)^3} L_n^v \ell_0^{v'}}. \end{aligned} \quad (2.9)$$

Remark 2.2

- (1) The events $A_{\bar{m}_1,1}$, $\bar{m}_1 \in T_{(n)}$, are decreasing and (2.8) immediately implies that

$$\begin{aligned} \mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} A_m^u\right] &\leq \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} A_{\bar{m}_1,1}^{u'}\right] \left(\mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} A_{\bar{m}_2,2}^{u'}\right] + 2e^{-2u' \frac{K}{(n+1)^3} L_n^v \ell_0^{v'}}\right). \end{aligned} \quad (2.8')$$

- (2) By the FKG Inequality (1.34) we see that

$$\mathbb{P}\left[\bigcap_{m \in T_{(n)}} A_m^u\right] \geq \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} A_{\bar{m}_1,1}^u\right] \mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} A_{\bar{m}_2,2}^u\right], \quad (2.10)$$

and a similar inequality holds for B_m^u , $m \in T_{(n+1)}$, $B_{\bar{m}_i,i}^u$, $\bar{m}_i \in T_{(n)}$, $i = 1, 2$.

We provide a brief outline of the proof of Theorem 2.1. When bounding the left-hand side of (2.8), (2.9), we have to cope with the mutual dependence between what happens in the union of boxes $\tilde{C}_{\bar{m}_1, \mathcal{T}_1}$ (of size $10L_0$) and in the union of boxes $\tilde{C}_{\bar{m}_2, \mathcal{T}_2}$ (of size $10L_0$) due to the presence of trajectories in the interlacement touching both collections. For this purpose we work in the above renormalization step with the two levels $0 < u' < u$. The sprinkling of additional trajectories corresponding to the increase from u' to u , is supposed to dominate for the most part interactions taking place at level u' . Specifically we consider two disjoint balls U_1, U_2 of radius cL_{n+1} , with c small, containing the respective collections of boxes, see (2.13), and we keep track of excursions of the trajectories in the interlacement point process at level u' between some region W , to be later determined, containing the two

collections of boxes of size $10L_0$, and the complement of $U = U_1 \cup U_2$. We collect all excursions generated by trajectories with labels at most u' that do reenter W after leaving U , see (2.42), and dominate with high probability this (non-Poissonian) point process in terms of the collection of excursions of trajectories with labels between u' and u that do not come back to W after leaving U . This domination step involves controls on the entrance distribution in W , which rely on the Harnack inequality from Lemma 1.2, see Lemma 2.3, and on a coupling lemma for the two point processes, see Lemma 2.4. We then choose the set W : a “large W ” offers a high success probability for the domination in the coupling, however requires that u and u' are sufficiently far apart (as a function of the “size of W ”), cf. (2.54). We optimize so that domination occurs with high probability, but u and u' remain close enough, cf. (2.58). Finally the coupling combined with the monotone character of the events in the left-hand side of (2.8), (2.9), enables us to derive the upper bounds (2.8), (2.9), see (2.65)–(2.68).

Proof of Theorem 2.1 We consider $n \geq 0$, $0 < u' < u$, $\mathcal{T} \in \Lambda_{n+1}$, and write for simplicity x_1, x_2 and \tilde{C}_1, \tilde{C}_2 , dropping the subscript \mathcal{T} in (2.2), when $m = 1, 2 \in T_{(1)}$. We then define

$$\hat{C}_i = \bigcup_{m \in T_{(n)}} \tilde{C}_{m, T_i}, \quad \text{for } i = 1, 2, \text{ and } V = \hat{C}_1 \cup \hat{C}_2. \quad (2.11)$$

In other words \hat{C}_1 and \hat{C}_2 are the respective unions of boxes of size of order L_0 corresponding to the respective descendants of 1 and 2 at the bottom scale. In particular we see by (2.3) that

$$\hat{C}_i \subseteq \tilde{C}_i, \quad \text{for } i = 1, 2. \quad (2.12)$$

Moreover by (2.1), (2.4), the following disjoint union

$$U = U_1 \cup U_2, \quad \text{where } U_i = B\left(x_i, \frac{L_{n+1}}{1000}\right), \quad i = 1, 2, \quad (2.13)$$

is such that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$, and

$$\tilde{C}_i \subseteq B\left(x_i, \frac{L_{n+1}}{2000M}\right) \stackrel{\text{def}}{=} \tilde{B}_i \subseteq U_i, \quad \text{for } i = 1, 2, \quad (2.14)$$

where M is determined in (2.36) below, and such that $1 \leq M \leq \ell_0/(2 \times 10^4)$. We then introduce a set W such that

$$V \subseteq W \subseteq \tilde{B}_1 \cup \tilde{B}_2 \subseteq U, \quad (2.15)$$

as well as the sequence $R_k, D_k, k \geq 1$ of successive returns of the walk on E to W and departures from U :

$$\begin{aligned} R_1 &= H_W, & D_1 &= T_U \circ \theta_{R_1} + R_1, & \text{and by induction} \\ R_{k+1} &= R_1 \circ \theta_{D_k} + D_k, & D_{k+1} &= D_1 \circ \theta_{D_k} + D_k, & \text{for } k \geq 1. \end{aligned} \quad (2.16)$$

Our goal is to keep track of excursions between W and ∂U for the paths of the random interlacement entering W , and in essence dominate the effect of paths with labels at most u' that reenter W for a second time, after leaving U , in terms of paths with labels between u' and u , that never reenter W after leaving U . Optimizing on W will then lead to the choice of W in (2.58) below.

We thus introduce the Poisson point processes on W^+ , (see (1.27) for notation),

$$\begin{aligned} \zeta'_\ell &= 1\{R_\ell < \infty = R_{\ell+1}\} \mu_{W,u'}, & \text{for } \ell \geq 1, \\ \zeta_\ell^* &= 1\{R_\ell < \infty = R_{\ell+1}\} \mu_{W,u',u}, & \text{for } \ell \geq 1. \end{aligned} \quad (2.17)$$

By (1.27) we see that

$$\zeta'_\ell, \ell \geq 1, \zeta_1^* \text{ are independent Poisson point processes on } W^+, \quad (2.18)$$

and their respective intensity measures are

$$\begin{aligned} \xi'_\ell &= u' 1\{R_\ell < \infty = R_{\ell+1}\} P_{e_W}, & \ell \geq 1, \\ \xi_1^* &= (u - u') 1\{R_1 < \infty = R_2\} P_{e_W}. \end{aligned} \quad (2.19)$$

Note also that the Poisson point processes on W^+

$$1\{H_{\widehat{C}_1} < \infty\} \mu_{W,u} \quad \text{and} \quad 1\{H_{\widehat{C}_1} = \infty\} (\zeta'_1 + \zeta_1^*) \text{ are independent.} \quad (2.20)$$

We then denote by \mathcal{C} the countable set of excursions from W to ∂U :

$$\begin{aligned} \mathcal{C} &= \left\{ \pi = (\pi(i))_{0 \leq i \leq N}, \quad \text{finite path, such that } \pi(0) \in W, \pi(N) \in \partial U, \text{ and} \right. \\ &\quad \left. \pi(i) \in U, \text{ for } 0 \leq i < N \right\}, \end{aligned} \quad (2.21)$$

and with ϕ_ℓ , when $\ell \geq 1$, the map from $\{R_\ell < \infty = R_{\ell+1}\} \subseteq W^+$ into \mathcal{C}^ℓ such that:

$$\begin{aligned} w &\rightarrow \phi_\ell(w) = (w_1, \dots, w_\ell), & \text{where} \\ w_k(\cdot) &= (X_{R_k + \cdot}(w))_{0 \leq \cdot \leq D_k(w) - R_k(w)}, & 1 \leq k \leq \ell. \end{aligned} \quad (2.22)$$

The ζ'_ℓ in (2.17) can be viewed as point processes on $\{R_\ell < \infty = R_{\ell+1}\} (\subseteq W^+)$, and ζ_1^* can be viewed as a point process on $\{R_1 < \infty = R_2\}$. We then

define

$$\begin{aligned} \tilde{\zeta}'_\ell & \text{ the image of } \zeta'_\ell \text{ under } \phi_\ell, \text{ for } \ell \geq 1, \text{ and} \\ \tilde{\zeta}_1^* & \text{ the image of } \zeta_1^* \text{ under } \phi_1, \end{aligned} \quad (2.23)$$

so that

$$\begin{aligned} \tilde{\zeta}'_\ell, \ell \geq 1, \tilde{\zeta}_1^* & \text{ are independent Poisson point processes with intensity} \\ \text{measures } \tilde{\xi}'_\ell, \ell \geq 1, \tilde{\xi}_1^*, & \text{ which are respective images under } \phi_\ell, \ell \geq 1, \text{ and} \\ \phi_1 \text{ of } \xi'_\ell, \ell \geq 1, \text{ and } \xi_1^*. & \end{aligned} \quad (2.24)$$

The following lemma plays a crucial role for the domination procedure, i.e. the sprinkling technique, we are currently setting into place. It brings into play the control of Harnack constants in $d(\cdot, \cdot)$ -balls from Lemma 1.2, together with some specific connectivity properties of the interior boundary of $d(\cdot, \cdot)$ -balls owing to the special structure of $E = G \times \mathbb{Z}$. As mentioned at the beginning of the section when $E = \mathbb{Z}^{d+1}$, $d \geq 2$, our results also hold when we instead work with the more common sup-norm distance and the corresponding balls, possibly adapting constants.

Lemma 2.3 ($\ell_0 \geq c$)

For any $\tilde{W} \subseteq B(x_1, L_{n+1}/2000) \cup B(x_2, L_{n+1}/2000)$, $x \in \partial U \cup \partial_{\text{int}} U$, $x' \in \tilde{W}$, one has

$$c_2' L_{n+1}^{-\nu} e_{\tilde{W}}(x') \leq P_x[H_{\tilde{W}} < \infty, X_{H_{\tilde{W}}} = x'] \leq c_2 L_{n+1}^{-\nu} e_{\tilde{W}}(x'). \quad (2.25)$$

Proof From the inclusion $\tilde{W} \subseteq U$, see (2.13), we have the classical sweeping identity, (for instance resulting from (1.27), (1.29)):

$$e_{\tilde{W}}(x') = P_{e_U}[H_{\tilde{W}} < \infty, X_{H_{\tilde{W}}} = x'], \quad (2.26)$$

so that

$$\begin{aligned} \text{cap}(U) \inf_{x \in \partial_{\text{int}} U} P_x[H_{\tilde{W}} < \infty, X_{H_{\tilde{W}}} = x'] \\ \leq e_{\tilde{W}}(x') \leq \text{cap}(U) \sup_{x \in \partial_{\text{int}} U} P_x[H_{\tilde{W}} < \infty, X_{H_{\tilde{W}}} = x']. \end{aligned} \quad (2.27)$$

Observe that the function $f(x) = P_x[H_{\tilde{W}} < \infty, X_{H_{\tilde{W}}} = x']$ is a non-negative function, which is harmonic in \tilde{W}^c . Now $\tilde{W} \subseteq B(x_1, \frac{L_{n+1}}{2000}) \cup B(x_2, \frac{L_{n+1}}{2000})$ and note that by (2.4), (2.13), (2.14) one has:

$$\begin{aligned} d(\partial_{\text{int}} U, B(x_1, L_{n+1}/2000) \cup B(x_2, L_{n+1}/2000)) \\ \geq \frac{L_{n+1}}{1000} - \frac{L_{n+1}}{2000} - 1 > \frac{L_{n+1}}{5000}. \end{aligned} \quad (2.28)$$

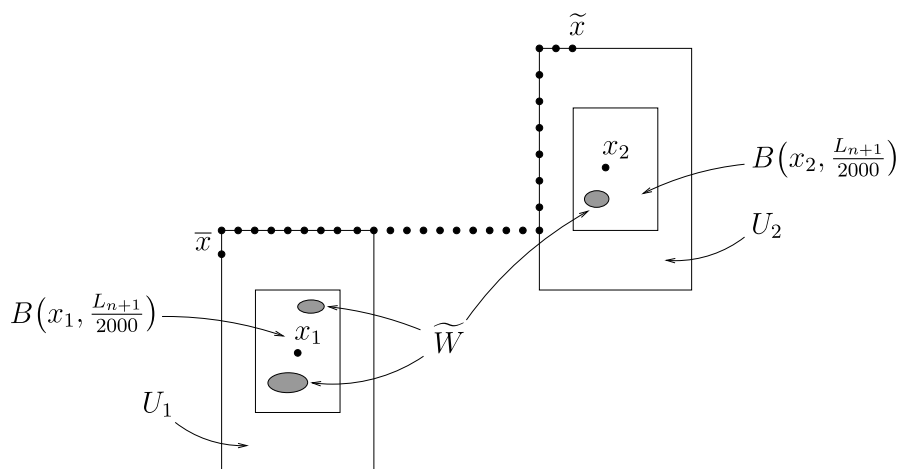


Fig. 1 A schematic illustration of a possible sequence $\bar{x}(i)$, $0 \leq i \leq \bar{c}$, corresponding to the black dots

Moreover $\partial_{\text{int}} U = \partial_{\text{int}} U_1 \cup \partial_{\text{int}} U_2$, and for $i = 1, 2$,

any two points on $\partial_{\text{int}} U_i$ can be linked by a path in $\partial_{\text{int}} U_i$ concatenation of at most three paths, which are either “horizontal” with at most $c L_{n+1}$ steps or “vertical” with at most $c L_{n+1}^{\beta/2}$ steps. (2.29)

The above statement is a straightforward consequence from the fact that for $x = (y, z)$ in E and $R \geq 1$ one has the identity

$$\begin{aligned} \partial_{\text{int}} B(x, R) &= B_G(y, R) \times (z + \{ -[R^{\beta/2}], [R^{\beta/2}] \}) \\ &\cup \partial_{\text{int}} B_G(y, R) \times (z + [-[R^{\beta/2}], [R^{\beta/2}]]). \end{aligned}$$

Note that $d(x_1, x_2) \leq 20L_{n+1}$ due to (2.2), (2.3). It now follows that there is an integer constant \bar{c} such that

for any two points \bar{x}, \tilde{x} in $\partial_{\text{int}} U$, one can construct $\bar{x}(i)$, $0 \leq i \leq \bar{c}$ (see Fig. 1), in $\partial_{\text{int}} U \cup U^c$, such that $\bar{x}(0) = \bar{x}$, $\bar{x}(\bar{c}) = \tilde{x}$, and

$$d(\bar{x}(i), \bar{x}(i+1)) \leq L \stackrel{\text{def}}{=} L_{n+1}/(10^4 c_0). \quad (2.30)$$

For each i , the function f is non-negative on $\overline{B(\bar{x}(i), c_0 L)}$ and harmonic in $B(\bar{x}(i), c_0 L)$ ($\subseteq \tilde{W}^c$) due to (2.28), (2.30). Applying Lemma 1.2 and a chaining argument, we see that:

$$\sup_{x \in \partial_{\text{int}} U} P_x[H_{\tilde{W}} < \infty, X_{H_{\tilde{W}}} = x'] \leq c \inf_{x \in \partial_{\text{int}} U} P_x[H_{\tilde{W}} < \infty, X_{H_{\tilde{W}}} = x']. \quad (2.31)$$

Moreover we know by (1.26), (1.22) that

$$c L_{n+1}^v \leq \text{cap}(U) \leq c' L_{n+1}^v. \quad (2.32)$$

Coming back to (2.27) we deduce (2.25) for $x \in \partial_{\text{int}} U$ and $x' \in \tilde{W}$. The extension to $x \in \partial U$ immediately follows thanks to the ellipticity condition satisfied by the walk, see above (1.6). \square

The next objective is to derive an upper bound on the intensity measures $\tilde{\xi}'_\ell$, $\ell \geq 2$, and a lower bound on $\tilde{\xi}_1^*$, see (2.40), (2.41) below. We first note that for $\ell \geq 2$, $w_1, \dots, w_\ell \in \mathcal{C}$, one has with the convention $R_0 = D_0 = 0$, as well as the notation $\bar{e}_W = e_W / \text{cap}(W)$,

$$\begin{aligned} & \tilde{\xi}'_\ell(w_1, \dots, w_\ell) \\ &= u' P_{e_W} [R_\ell < \infty, (X_{R_k+\cdot})_{0 \leq \cdot \leq D_k - R_k} = w_k(\cdot), 1 \leq k \leq \ell, R_{\ell+1} = \infty] \\ & \stackrel{\text{strong Markov}}{\leq} u' E_{e_W} [R_\ell < \infty, (X_{R_k+\cdot})_{0 \leq \cdot \leq D_k - R_k} = w_k(\cdot), 1 \leq k < \ell, \\ & \quad \text{property} \quad P_{X_{R_\ell}} [(X_\cdot)_{0 \leq \cdot \leq T_U} = w_\ell(\cdot)]] \\ &= u' E_{e_W} [D_{\ell-1} < \infty, (X_{R_k+\cdot})_{0 \leq \cdot \leq D_k - R_k} = w_k(\cdot), 1 \leq k < \ell, \\ & \quad E_{X_{D_{\ell-1}}} [H_W < \infty, P_{X_{H_W}} [(X_\cdot)_{0 \leq \cdot \leq T_U} = w_\ell(\cdot)]]] \\ & \stackrel{(2.25)}{\leq} u' \frac{c_2}{L_{n+1}^v} \text{cap}(W) E_{e_W} [D_{\ell-1} < \infty, (X_{R_k+\cdot})_{0 \leq \cdot \leq D_k - R_k} = w_k, \\ & \quad 1 \leq k \leq \ell - 1] \\ & \quad \times P_{\bar{e}_W} [(X_\cdot)_{0 \leq \cdot \leq T_U} = w_\ell(\cdot)], \end{aligned} \quad (2.33)$$

and by induction

$$\leq u' \text{cap}(W) \left(c_2 \frac{\text{cap}(W)}{L_{n+1}^v} \right)^{\ell-1} \prod_{k=1}^{\ell} \Gamma(w_k),$$

where Γ is the law on \mathcal{C} of $(X_\cdot)_{0 \leq \cdot \leq T_U}$ under $P_{\bar{e}_W}$:

$$\Gamma(\cdot) = P_{\bar{e}_W} [(X_\cdot)_{0 \leq \cdot \leq T_U} = \cdot]. \quad (2.34)$$

As for the lower bound on $\tilde{\xi}_1^*$, we note that for $w \in \mathcal{C}$ one has

$$\tilde{\xi}_1^*(w) = (u - u') P_{e_W} [(X_\cdot)_{0 \leq \cdot \leq T_U} = w(\cdot), H_W \circ \theta_{T_U} = \infty]. \quad (2.35)$$

Observe that for $x \in \partial U$ we have by (2.14), (2.15), and (2.25)

$$P_x[H_W < \infty] \leq \frac{c_2}{L_{n+1}^v} \text{cap}(W) \stackrel{(1.26)}{\leq} \frac{c}{L_{n+1}^v} \left(\frac{L_{n+1}}{2000M} \right)^v \leq (2e)^{-1},$$

when we choose $M = c_3 \geq 1$,

(2.36)

and we assume $\ell_0 \geq 2 \cdot 10^4 c_3$, so that $1 \leq M \leq \ell_0 / (2 \cdot 10^4)$. Coming back to (2.35) we see applying the strong Markov property at time T_U that for $w \in \mathcal{C}$:

$$\tilde{\xi}_1^*(w) \geq \frac{(u - u')}{2} \text{cap}(W) \Gamma(w). \quad (2.37)$$

We then introduce the notation

$$\lambda'_W = u' \text{cap}(W), \quad \beta_W = \frac{c_2 \text{cap}(W)}{L_{n+1}^v} \left(\leq \frac{1}{2e}, \text{ see (2.36)} \right), \quad (2.38)$$

as well as

$$\lambda_W^* = \frac{(u - u')}{2} \text{cap}(W). \quad (2.39)$$

We have thus shown that

$$\tilde{\xi}'_\ell \leq \lambda'_W \beta_W^{\ell-1} \Gamma^{\otimes \ell}, \quad \text{for } \ell \geq 2, \text{ and} \quad (2.40)$$

$$\tilde{\xi}_1^* \geq \lambda_W^* \Gamma. \quad (2.41)$$

We now come to the construction of a coupling that will provide the domination argument hinted at in the comment below (2.16). We denote by s_ℓ the map which sends a finite point measure m on \mathcal{C}^ℓ onto a finite point measure on \mathcal{C} via:

$$s_\ell(m) = \sum_{k=1}^{\ell} p_k^\ell \circ m,$$

where p_k^ℓ stands for the k -th canonical coordinate on \mathcal{C}^ℓ , for $1 \leq k \leq \ell$. In other words $s_\ell(m) = \sum_{i=1}^N \delta_{w_1^i} + \cdots + \delta_{w_\ell^i}$, when $m = \sum_{i=1}^N \delta_{(w_1^i, \dots, w_\ell^i)}$. We want to construct a coupling of the finite point measure on \mathcal{C} :

$$\tilde{\sigma}' = \sum_{\ell \geq 2} s_\ell(\tilde{\xi}'_\ell), \quad (2.42)$$

which collects all excursions from W to ∂U induced by trajectories in the support of $\mu_{W, \underline{w}'}$ having at least once exited U and reentered W , with the finite point measure $\tilde{\xi}_1^*$, so that with high probability $\tilde{\xi}_1^*$ dominates $\tilde{\sigma}'$. Incidentally

let us mention that $\tilde{\sigma}'$ is not a Poisson point measure, see Remark 2.5 (1) below. This feature makes the domination argument more delicate.

With this in mind we consider an auxiliary probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ endowed with a collection of independent Poisson variables N'_ℓ , $\ell \geq 2$, and with a Poisson variable N_1^* , with respective intensities,

$$\lambda'_\ell = \lambda'_W \beta_W^{\ell-1}, \quad \ell \geq 2, \quad \text{and} \quad (2.43)$$

$$\lambda_1^* = \lambda_W^*, \quad (2.44)$$

and with an independent collection of i.i.d. \mathcal{C} -valued random variables γ_i , $i \geq 1$, with distribution Γ , cf. (2.34). We then define

$$N' = \sum_{\ell \geq 2} \ell N'_\ell. \quad (2.45)$$

By (2.38), we see that N' has finite expectation, (actually N' is distributed as a certain Poisson compound distribution, see below (2.51)), and we define the finite point processes on \mathcal{C} :

$$\Sigma' = \sum_{1 \leq i \leq N'} \delta_{\gamma_i}, \quad \Sigma_1^* = \sum_{1 \leq i \leq N_1^*} \delta_{\gamma_i}. \quad (2.46)$$

Lemma 2.4 (Coupling Lemma)

One can construct on some probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$ variables N' , N_1^ , Σ' , Σ_1^* as above, and $\bar{\sigma}'$ distributed as $\tilde{\sigma}'$, and $\bar{\zeta}_1^*$ distributed as $\tilde{\zeta}_1^*$ so that*

$$\bar{\sigma}' \leq \Sigma' \quad \text{and} \quad \Sigma_1^* \leq \bar{\zeta}_1^*, \quad (2.47)$$

and in particular

$$\bar{\mathbb{P}}\text{-a.s. on } \{N' \leq N_1^*\}, \quad \bar{\sigma}' \leq \bar{\zeta}_1^*. \quad (2.48)$$

Proof Observe that Σ' has the same distribution as $\sum_{\ell \geq 2} s_\ell(\bar{\zeta}_\ell)$, where $\bar{\zeta}_\ell$, $\ell \geq 2$, are independent Poisson point processes on \mathcal{C}^ℓ with respective intensities $\lambda'_\ell \Gamma^{\otimes \ell} \geq \tilde{\xi}'_\ell$, due to (2.40), (2.43). We construct a coupling of the $\tilde{\zeta}'_\ell$, $\ell \geq 2$, and the $\bar{\zeta}_\ell$, $\ell \geq 2$, say by thinning the $\bar{\zeta}_\ell$, $\ell \geq 2$, so that $\bar{\zeta}_\ell \geq \tilde{\zeta}'_\ell$, for each $\ell \geq 2$. This permits after “enlarging the probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ ”, to construct a $\bar{\sigma}'$ with same distribution as $\tilde{\sigma}'$ and such that $\Sigma' \geq \bar{\sigma}'$.

Then note that Σ_1^* is a Poisson point process on \mathcal{C} with intensity measure $\lambda_1^* \Gamma \leq \tilde{\xi}_1^*$ by (2.41), (2.44). We can then construct, by “further enlarging” the probability space, a $\bar{\zeta}_1^*$ having same law as $\tilde{\zeta}_1^*$, such that $\Sigma_1^* \leq \bar{\zeta}_1^*$. This yields (2.47) and (2.48) is an immediate consequence. \square

The next task is to bound the probability of the “bad event” $\{N_1^* < N'\}$, cf. (2.48), and by appropriately selecting W in (2.58) obtain the bound (2.59). We thus note that

$$\overline{\mathbb{P}}[N_1^* < N'] \leq \overline{\mathbb{P}}\left[N_1^* \leq \frac{\lambda_W^*}{2}\right] + \overline{\mathbb{P}}\left[N' > \frac{\lambda_W^*}{2}\right]. \quad (2.49)$$

By classical bounds on the tail of a Poisson variable of intensity λ_W^* we have:

$$\overline{\mathbb{P}}\left[N_1^* \leq \frac{\lambda_W^*}{2}\right] \leq e^{-c\lambda_W^*}. \quad (2.50)$$

As for the last term of (2.49), the exponential Chebyshev inequality yields

$$\overline{\mathbb{P}}\left[N' > \frac{\lambda_W^*}{2}\right] \leq \exp\left\{-a \frac{\lambda_W^*}{2}\right\} E\overline{\mathbb{P}}[e^{aN'}], \quad \text{for } a > 0. \quad (2.51)$$

We then observe that

$$E\overline{\mathbb{P}}[e^{aN'}] \stackrel{(2.45)}{=} \exp\left\{\sum_{\ell \geq 2} \lambda'_\ell (e^{a\ell} - 1)\right\} = \exp\left\{\lambda'_W \sum_{\ell \geq 2} \beta_W^{\ell-1} (e^{a\ell} - 1)\right\}.$$

This formula incidentally shows that N' has the law of a compound Poisson variable with compounding distribution which is that of a geometric variable with parameter $1 - \beta_W$ conditioned to be at least 2. Note that $E[e^{aN'}] = \infty$, when $e^a \geq \beta_W^{-1}$, but that

$$E\overline{\mathbb{P}}[e^{aN'}] \leq \exp\left\{\lambda'_W \beta_W \frac{e^{2a}}{1 - \beta_W e^a}\right\}, \quad \text{when } \beta_W e^a < 1. \quad (2.52)$$

Coming back to (2.51), we have shown that

$$\overline{\mathbb{P}}\left[N' > \frac{\lambda_W^*}{2}\right] \leq \exp\left\{-a \frac{\lambda_W^*}{2} + \lambda'_W \beta_W \frac{e^{2a}}{1 - \beta_W e^a}\right\}, \quad \text{for } \beta_W e^a < 1. \quad (2.53)$$

Choosing $a = 1$, we know from (2.38) that $2e\beta_W \leq 1$, so that when $\frac{\lambda_W^*}{4} \geq 2\beta_W e^2 \lambda'_W$ holds, one finds that $\overline{\mathbb{P}}[N' > \frac{\lambda_W^*}{2}] \leq \exp\{-\frac{\lambda_W^*}{4}\}$. Thus coming back to (2.49) we have shown that for any W such that $V \subseteq W \subseteq \bigcup_{i=1}^2 B(x_i, L_{n+1}/(2000c_3)) = \tilde{B}_1 \cup \tilde{B}_2$, cf. (2.15), (2.36), (2.39), one has

$$\overline{\mathbb{P}}[N_1^* < N'] \leq 2 \exp\{-c \lambda_W^*\}, \quad \text{when } (u - u') \geq 16e^2 c_2 \frac{\text{cap}(W)}{L_{n+1}^v} u'. \quad (2.54)$$

We will now select the set W . We want on the one hand $\text{cap}(W)$ large, to take advantage of the left-hand inequality in (2.54), and on the other hand not too

large, so that we can pick u, u' close enough (in our renormalization scheme, the sequence of levels will need to converge).

Due to (1.26) and (2.11), we have

$$\text{cap}(V) \leq \tilde{c} 2^{n+1} L_0^v, \quad (2.55)$$

as well as

$$\text{cap}(\tilde{B}_1 \cup \tilde{B}_2) \geq \bar{c} L_{n+1}^v = \bar{c} \ell_0^{(n+1)v} L_0^v. \quad (2.56)$$

Observe that for $A \subset\subset E$, $x \in E$, one has, cf. (1.22)

$$\text{cap}(A) \leq \text{cap}(A \cup \{x\}) \leq \text{cap}(A) + c_*. \quad (2.57)$$

Thus when $\ell_0 \geq c(K, v')$, one can find W such that $V \subseteq W \subseteq \tilde{B}_1 \cup \tilde{B}_2$ and

$$\frac{1}{2} \frac{\sqrt{K}}{(n+1)^{3/2}} L_n^v \ell_0^{v''} \leq \text{cap}(W) \leq 2 \frac{\sqrt{K}}{(n+1)^{3/2}} L_n^v \ell_0^{v''}, \quad \text{where } v'' = \frac{v+v'}{2}. \quad (2.58)$$

Indeed, when $\ell_0 \geq c(K, v')$, with the same constants as in (2.55)–(2.57), one has for all $n \geq 0$, $\tilde{c} 2^{n+1} L_0^v \leq \frac{\sqrt{K}}{(n+1)^{3/2}} \ell_0^{nv} L_0^v \ell_0^{v''} < \bar{c} \ell_0^{(n+1)v} L_0^v$, as well as $\frac{\sqrt{K}}{(n+1)^{3/2}} \ell_0^{nv} L_0^v \ell_0^{v''} > 2c_*$.

So when $\ell_0 \geq c(K, v')$, with the above choice of W and (2.54), we have shown that

$$\begin{aligned} & \text{when } u - u' \geq c_1 \sqrt{K}/(n+1)^{3/2} \ell_0^{-(v-v'')} u', \quad \text{then} \\ & \overline{\mathbb{P}}[N_1^* < N'] \leq 2 \exp \left\{ -c(u - u') \frac{\sqrt{K}}{(n+1)^{3/2}} L_n^v \ell_0^{v''} \right\} \\ & \leq 2 \exp \left\{ -2u' \frac{K}{(n+1)^3} L_n^v \ell_0^{v'} \right\} \\ & \quad (\text{since } 2v'' - v = v' \text{ and choosing } c_1 \text{ large enough}). \end{aligned} \quad (2.59)$$

We now introduce the random finite subsets of $V(\subseteq W)$:

$$\begin{aligned} \mathcal{I}'_\ell &= V \cap \left(\bigcup_{(w_1, \dots, w_\ell) \in \text{Supp } \tilde{\zeta}'_\ell} \text{range}(w_1) \cup \dots \cup \text{range}(w_\ell) \right), \\ & \quad \text{for } \ell \geq 1, \end{aligned} \quad (2.60)$$

$$\mathcal{I}_1^* = V \cap \left(\bigcup_{w \in \text{Supp } \tilde{\zeta}_1^*} \text{range}(w) \right). \quad (2.61)$$

By (2.23), (2.24) we see that

$$\begin{aligned} &\text{the random subsets } \mathcal{I}'_\ell, \ell \geq 1, \mathcal{I}_1^* \text{ are independent,} \\ &\mathcal{I}^{u'} \cap V = \bigcup_{\ell \geq 1} \mathcal{I}'_\ell, \quad \text{and} \quad \mathcal{I}^u \cap V \supseteq \mathcal{I}_1^*. \end{aligned} \quad (2.62)$$

We also note that

$$\mathcal{I}^u \cap \widehat{C}_1 = \widehat{C}_1 \cap \left(\bigcup_{w \in \text{Supp}(1_{\{H_{\widehat{C}_1} < \infty\}} \mu_{W,u})} \text{range}(w) \right),$$

and that

$$(\mathcal{I}'_1 \cup \mathcal{I}_1^*) \cap \widehat{C}_2 = \widehat{C}_2 \cap \left(\bigcup_{w \in \text{Supp}(1_{\{H_{\widehat{C}_1} = \infty\}} (\zeta'_1 + \zeta_1^*))} \text{range}(w) \right).$$

By (2.20) we thus see that

$$\mathcal{I}^u \cap \widehat{C}_1 \quad \text{and} \quad (\mathcal{I}'_1 \cup \mathcal{I}_1^*) \cap \widehat{C}_2 \text{ are independent.} \quad (2.63)$$

By even easier arguments we also see that

$$(\mathcal{I}'_1 \cup \mathcal{I}_1^*) \cap \widehat{C}_1 \quad \text{and} \quad (\mathcal{I}'_1 \cup \mathcal{I}_1^*) \cap \widehat{C}_2 \text{ are independent.} \quad (2.64)$$

We now prove our main claims (2.8), (2.9). We recall the notation from (1.35). Since the $A_m, m \in I_{(n+1)}$, are decreasing and \mathcal{T} -adapted, cf. (2.5), we see that

$$\begin{aligned} &\mathbb{P} \left[\bigcap_{m \in T_{(n+1)}} A_m^u \right] \\ &\stackrel{(2.6)}{=} \mathbb{P} \left[\bigcap_{\overline{m}_1 \in T_{(n)}} A_{\overline{m}_1,1}^u \cap \bigcap_{\overline{m}_2 \in T_{(n)}} A_{\overline{m}_2,2}^u \right] \\ &\leq \mathbb{P} \left[\bigcap_{\overline{m}_1 \in T_{(n)}} A_{\overline{m}_1,1}(\mathcal{I}^u \cap \widehat{C}_1) \cap \bigcap_{\overline{m}_2 \in T_{(n)}} A_{\overline{m}_2,2}((\mathcal{I}'_1 \cup \mathcal{I}_1^*) \cap \widehat{C}_2) \right] \\ &\stackrel{(2.63)}{=} \mathbb{P} \left[\bigcap_{\overline{m}_1 \in T_{(n)}} A_{\overline{m}_1,1}^u \right] \mathbb{P} \left[\bigcap_{\overline{m}_2 \in T_{(n)}} A_{\overline{m}_2,2}(\mathcal{I}'_1 \cup \mathcal{I}_1^*) \right]. \end{aligned} \quad (2.65)$$

Note that by (2.42), (2.60) we have

$$\bigcup_{\ell \geq 2} \mathcal{I}'_\ell = V \cap \left(\bigcup_{w \in \text{Supp } \tilde{\sigma}'} \text{range}(w) \right), \quad (2.66)$$

and it follows from Lemma 2.4 and (2.62) that

$$\begin{aligned}
 & \mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} A_{\bar{m}_2, 2}(\mathcal{I}'_1 \cup \mathcal{I}_1^*)\right] \\
 & \leq \mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} A_{\bar{m}_2, 2}\left(\bigcup_{\ell \geq 1} \mathcal{I}'_\ell\right)\right] + \bar{\mathbb{P}}[N_1^* < N'] \\
 & \stackrel{(2.59), (2.62)}{\leq} \mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} A_{\bar{m}_2, 2}^{u'}\right] + 2 \exp\left\{-2u' \frac{K}{(n+1)^3} L_n^\nu \ell_0^{\nu'}\right\}, \\
 & \text{when } u - u' \geq c_1 \frac{\sqrt{K}}{(n+1)^{3/2}} \ell_0^{-\frac{(\nu-\nu')}{2}} u'. \tag{2.67}
 \end{aligned}$$

Inserting this inequality in (2.65) yields (2.8) (and of course (2.8') as well). We now turn to the proof of (2.9). Since the B_m , $m \in T_{(n+1)}$, are increasing \mathcal{T} -adapted, we find with Lemma 2.4 that

$$\begin{aligned}
 & \mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} B_m^{u'}\right] \\
 & \stackrel{(2.62)}{=} \mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} B_m\left(\bigcup_{\ell \geq 1} \mathcal{I}'_\ell\right)\right] \\
 & \stackrel{(2.62), (2.66)}{\leq} \mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} B_m(\mathcal{I}'_1 \cup \mathcal{I}_1^*)\right] + \bar{\mathbb{P}}[N_1^* < N'] \\
 & \stackrel{\text{Lemma 2.4}}{\leq} \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} B_{\bar{m}_1, 1}(\mathcal{I}'_1 \cup \mathcal{I}_1^*)\right] + \bar{\mathbb{P}}[N_1^* < N'] \\
 & \stackrel{(2.64)}{=} \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} B_{\bar{m}_1, 1}(\mathcal{I}'_1 \cup \mathcal{I}_1^*)\right] + \bar{\mathbb{P}}[N_1^* < N'] \\
 & \stackrel{(2.62)}{\leq} \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} B_{\bar{m}_1, 1}^u\right] + \bar{\mathbb{P}}[N_1^* < N]. \tag{2.68}
 \end{aligned}$$

The claim (2.9) now follows from (2.59), and this completes the proof of Theorem 2.1. \square

Remark 2.5

- (1) With the help of Lemma 2.3 and (2.36), one can also derive a companion lower bound to (2.33) or (2.40) showing that for $\ell \geq 2$,

$$\tilde{\xi}'_\ell \geq u' \operatorname{cap}(W) \left(c \frac{\operatorname{cap}(W)}{L_{n+1}^\nu}\right)^{\ell-1} \Gamma^{\otimes \ell}.$$

Combined with (2.40) we see that the total mass of $\tilde{\sigma}'$, see (2.42), has finite expectation, but that large enough exponential moments of the total mass of $\tilde{\sigma}'$ are divergent, (this is very much in line with the identity below (2.51)).

As a result one cannot hope to find a coupling of $\tilde{\sigma}'$ and $\tilde{\zeta}_1^*$ such that $\tilde{\zeta}_1^*$ globally dominates $\tilde{\sigma}'$, see Lemma 2.4.

Incidentally in the construction of the coupling from Lemma 2.4, one actually has flexibility in the choice of the joint distribution of N' and N_1^* . One might possibly take advantage of this feature to improve the bound (2.49), and as a result improve the quality of the remainder terms in (2.8), (2.9).

- (2) The renormalization step conducted above differs from what was done in [18, 19, 21]. In essence, the sprinkling technique we have employed here, aims at dominating for the most part, with the Poisson variable N_1^* , the “long range interaction terms”, which are accounted for in the variable N' of (2.45). The variable N' has a compound Poisson distribution, where the compounding law is supported on the set of integers at least equal to 2, see below (2.51). Quite naturally, the quality of our bounds deteriorates when the parameter of N_1^* becomes small. This explains why in place of choosing $W = V$ in (2.54), we instead pick W rather big, at the expense of assuming $u - u'$ not too small.
- (3) Although Theorem 2.1 will be general enough for our purpose here, let us note that the crucial Lemma 2.4 offers a domination statement that goes beyond asserting that $\bigcup_{\ell \geq 2} \mathcal{I}'_\ell$ is dominated with “high probability” by \mathcal{I}_1^* . For instance the proof of Theorem 2.1 yields similar inequalities as (2.8), (2.9) for decreasing events A_m , $m \in T_{(n+1)}$, or increasing events B_m , $m \in T_{(n+1)}$, pertaining to the occupancy of both sets of points and edges contained in $\tilde{C}_{m, \mathcal{T}}$. In place of (1.36) one naturally defines in this context A_m^u , or B_m^u , by the consideration of the set of points visited and edges crossed by the trajectories of the interlacement point process with labels at most u . The proof of Theorem 2.1 then yields.

Corollary 2.1' ($K > 0$, $0 < v' < v$)

When $\ell_0 \geq c(K, v')$, for all $n \geq 0$, $\mathcal{T} \in \Lambda_{n+1}$, for all in the above generalized sense \mathcal{T} -adapted collections A_m , $m \in T_{(n+1)}$, of decreasing events, respectively B_m , $m \in T_{(n+1)}$, of increasing events, and for all $0 < u' < u$ satisfying (2.7), the corresponding inequalities to (2.8), (2.9) hold.

We will now derive the key decoupling inequalities. Given $K > 0$, $0 < v' < v$, and $\ell_0 \geq c(K, v')$ such that Theorem 2.1 applies we define for any

$u_0 > 0$ the increasing and decreasing sequences of levels

$$\begin{aligned} u_n^+ &= \prod_{0 \leq k < n} \left(1 + c_1 \frac{\sqrt{K}}{(k+1)^{3/2}} \ell_0^{-\frac{(v-v')}{2}} \right) u_0, \\ u_n^- &= \prod_{0 \leq k < n} \left(1 + c_1 \frac{\sqrt{K}}{(k+1)^{3/2}} \ell_0^{-\frac{(v-v')}{2}} \right)^{-1} u_0, \end{aligned} \quad (2.69)$$

so that $u_n^+, n \geq 0$, is increasing, $u_n^-, n \geq 0$, is decreasing, and they satisfy

$$\begin{aligned} u_{n+1}^+ - u_n^+ &= c_1 \frac{\sqrt{K}}{(n+1)^{3/2}} \ell_0^{-\frac{(v-v')}{2}} u_n^+, \quad \text{and} \\ u_n^- - u_{n+1}^- &= c_1 \frac{\sqrt{K}}{(n+1)^{3/2}} \ell_0^{-\frac{(v-v')}{2}} u_{n+1}^-, \end{aligned}$$

for $n \geq 0$, as well as $u_0^+ = u_0 = u_0^-$.

These sequences also have positive finite limits respectively equal to

$$\begin{aligned} u_\infty^+ &= \prod_{k \geq 0} \left(1 + c_1 \frac{\sqrt{K}}{(k+1)^{3/2}} \ell_0^{-\frac{(v-v')}{2}} \right) u_0, \\ u_\infty^- &= \prod_{k \geq 0} \left(1 + c_1 \frac{\sqrt{K}}{(k+1)^{3/2}} \ell_0^{-\frac{(v-v')}{2}} \right)^{-1} u_0. \end{aligned} \quad (2.70)$$

We can now state and prove the main result of this section.

Theorem 2.6 (Decoupling Inequalities; $K > 0$, $0 < v' < v$, $\ell_0 \geq c(K, v')$)

For any $u_0 > 0$, $n \geq 0$, $\mathcal{T} \in \Lambda_n$, and all \mathcal{T} -adapted collections $A_m, m \in T_{(n)}$, of decreasing events on $\{0, 1\}^E$, respectively $B_m, m \in T_{(n)}$, of increasing events on $\{0, 1\}^E$, one has:

$$\mathbb{P} \left[\bigcap_{m \in T_{(n)}} A_m^{u_\infty^+} \right] \leq \mathbb{P} \left[\bigcap_{m \in T_{(n)}} A_m^{u_n^+} \right] \leq \prod_{m \in T_{(n)}} (\mathbb{P}[A_m^{u_0}] + \varepsilon(u_0)), \quad (2.71)$$

$$\mathbb{P} \left[\bigcap_{m \in T_{(n)}} B_m^{u_\infty^-} \right] \leq \mathbb{P} \left[\bigcap_{m \in T_{(n)}} B_m^{u_n^-} \right] \leq \prod_{m \in T_{(n)}} (\mathbb{P}[B_m^{u_0}] + \varepsilon(u_\infty^-)), \quad (2.72)$$

where we have set

$$\begin{aligned} \varepsilon(u) &= \frac{2e^{-KuL_0^v \ell_0^{v'}}}{1 - e^{-KuL_0^v \ell_0^{v'}}}, \\ \text{for } u > 0 \text{ (note that } v > 0 \rightarrow \frac{e^{-v}}{1-e^{-v}} \text{ is decreasing)}. \end{aligned} \quad (2.73)$$

Proof We begin with the proof of (2.72). The first inequality is immediate since $u_\infty^- \leq u_n^-$, and the events B_m are increasing. As for the second inequality, we first prove by induction on n that

$$\mathbb{P}\left[\bigcap_{m \in T_{(n)}} B_m^{u_n^-}\right] \leq \prod_{m \in T_{(n)}} \left(\mathbb{P}[B_m^{u_0}] + \sum_{0 \leq k < n} \left(2e^{-\frac{2K}{(k+1)^3} u_{k+1}^- L_k^v \ell_0^{v'}} \right)^{\frac{1}{2k+1}} \right). \quad (2.74)$$

The claim trivially holds when $n = 0$. If it is true for n , we find by (2.9) that for $\mathcal{T} \in \Lambda_{n+1}$, and B_m , $m \in T_{(n+1)}$, an increasing \mathcal{T} -adapted collection, one has, with hopefully obvious notation:

$$\begin{aligned} & \mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} B_m^{u_{n+1}^-}\right] \\ & \leq \mathbb{P}\left[\bigcap_{\bar{m}_1 \in T_{(n)}} B_{\bar{m}_1,1}^{u_n^-}\right] \mathbb{P}\left[\bigcap_{\bar{m}_2 \in T_{(n)}} B_{\bar{m}_2,2}^{u_n^-}\right] + 2e^{-\frac{2K}{(n+1)^3} u_{n+1}^- L_n^v \ell_0^{v'}} \\ & \stackrel{\text{induction}}{\leq} \prod_{m \in T_{(n+1)}} \left(\mathbb{P}[B_m^{u_0}] + \sum_{0 \leq k < n} \left(2e^{-\frac{2K}{(k+1)^3} u_{k+1}^- L_k^v \ell_0^{v'}} \right)^{\frac{1}{2k+1}} \right) \\ & \quad + 2e^{-\frac{2K}{(n+1)^3} u_{n+1}^- L_n^v \ell_0^{v'}} \\ & \leq \prod_{m \in T_{(n+1)}} \left(\mathbb{P}[B_m^{u_0}] + \sum_{0 \leq k < n+1} \left(2e^{-\frac{2K}{(k+1)^3} u_{k+1}^- L_k^v \ell_0^{v'}} \right)^{\frac{1}{2k+1}} \right), \quad (2.75) \end{aligned}$$

and this completes the proof of (2.74) by induction.

To finish the proof of (2.72), we now observe that when $\ell_0 \geq c(K, v')$, then for all $k \geq 0$, $(\frac{\ell_0^v}{2})^k \geq 2^{4k} \geq (k+1)^4$. Hence the series in the product in the right-hand side of (2.74) is smaller than

$$\sum_{k \geq 0} 2e^{-\frac{K}{(k+1)^3} u_\infty^- (\frac{\ell_0^v}{2})^k L_0^v \ell_0^{v'}} \leq 2 \sum_{k \geq 0} e^{-(k+1)K u_\infty^- L_0^v \ell_0^{v'}} = \varepsilon(u_\infty^-),$$

and this completes the proof of (2.72).

The proof of (2.71) is analogous. Instead of (2.74), one shows by induction on n , with the help of (2.8'), that

$$\mathbb{P}\left[\bigcap_{m \in T_{(n)}} A_m^{u_n^+}\right] \leq \prod_{m \in T_{(n)}} \left(\mathbb{P}[A_m^{u_0}] + \sum_{0 \leq k < n} \left(2e^{-\frac{2K}{(k+1)^3} u_k^+ L_k^v \ell_0^{v'}} \right)^{\frac{1}{2k}} \right), \quad (2.76)$$

and obtains (2.71) as a consequence, (in fact the argument below (2.75) yields that (2.71) even holds with $2K$ in place of K in the definition of $\varepsilon(u_0)$ in (2.73)). \square

Remark 2.7

- (1) As already mentioned, in the important special case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, (so that $\alpha = d$, $\beta = 2$), one can replace the distance $d(\cdot, \cdot)$ with the sup-norm distance $d_\infty(\cdot, \cdot)$ and the balls relative to $d(\cdot, \cdot)$ with balls relative to $d_\infty(\cdot, \cdot)$, in the above theorem and of course in the definition of Λ_n , and of \mathcal{T} -adapted collections when $\mathcal{T} \in \Lambda_n$.
- (2) As companions to (2.71), (2.72), the FKG-Inequality, see (1.34), implies that

$$\mathbb{P}\left[\bigcap_{m \in T(n)} A_m^{u_\infty^+}\right] \geq \prod_{m \in T(n)} \mathbb{P}[A_m^{u_\infty^+}], \quad (2.77)$$

and

$$\mathbb{P}\left[\bigcap_{m \in T(n)} B_m^{u_\infty^-}\right] \geq \prod_{m \in T(n)} \mathbb{P}[B_m^{u_\infty^-}]. \quad (2.78)$$

Theorem 2.6 provides an upper bound for the expressions in the left-hand side of (2.77), (2.78). In a sense it offers a partial substitute for the BK Inequality, which plays a key role for Bernoulli percolation, cf. [13], but not for interlacement percolation, so far, see Remark 1.5 (3).

3 Cascading events

We want to apply the decoupling inequalities of Theorem 2.6 to control the probability of certain events on $\{0, 1\}^E$, which pertain to the trace of the interlacement \mathcal{I}^u in a large box. For this purpose the cascading property plays an important role. In essence, it enables us to cover such an event concerning the state of \mathcal{I}^u in a ball of size of order L_n , with a not too large family of events, which come each as intersections of 2^n events depending on the respective traces of \mathcal{I}^u in well-separated balls of size of order L_0 . The decoupling inequalities of Theorem 2.6 will yield upper bounds on the probability of such intersections. These bounds will compete against the combinatorial complexity of the family of dyadic tree embeddings used to cover the original event. In Proposition 3.2 we present two examples of cascading families of events, which play an important role in Sects. 4 and 5. We discuss in Remark 3.3 another example in the case of \mathbb{Z}^{d+1} , $d \geq 2$, which is due to Teixeira [27], as well as a modification of this example adapted to the general context of the present work. The main consequences of the cascading property and the decoupling inequalities appear in Theorem 3.4, as well as Corollary 3.5 and 3.7.

In the special case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, the distance $d(\cdot, \cdot)$ and the balls $B(x, r)$ can be replaced with the sup-norm distance $d_\infty(\cdot, \cdot)$ and the corresponding balls $B_\infty(\cdot, \cdot)$.

Definition 3.1 A family $\mathcal{G} = (G_{x,L})_{x \in E, L \geq 1 \text{ integer}}$ of events on $\{0, 1\}^E$ has the cascading property (or cascades) with complexity at most $\lambda > 0$, when

$$G_{x,L} \text{ is } \sigma(\Psi_{x'}, x' \in B(x, 10L))\text{-measurable for each } x \in E, L \geq 1, \quad (3.1)$$

(the notation Ψ_x is defined above (1.34)), and for each ℓ multiple of 100, $x \in E, L \geq 1$, there exists $\Lambda \subseteq E$, such that

$$\Lambda \subseteq B(x, 9\ell L), \quad (3.2)$$

$$|\Lambda| \leq c(\mathcal{G}, \lambda) \ell^\lambda, \quad (3.3)$$

$$G_{x,\ell L} \subseteq \bigcup_{x', x'' \in \Lambda; d(x', x'') \geq \frac{\ell}{100} L} G_{x',L} \cap G_{x'',L}. \quad (3.4)$$

When the family of events depends on an additional parameter, we say that it has the *uniform cascading property* (or *cascades uniformly*) with complexity at most λ , when for each fixed value of the parameter, the cascading property with complexity at most λ holds, and in addition the constant in (3.3) can be chosen uniformly in the parameter.

We will now provide examples of such families, see Proposition 3.2 and Remark 3.3 below. The first two examples play an important role, respectively in Sects. 4 and 5. The first example corresponds to the family $\mathcal{A} = (A_{x,L})_{x \in E, L \geq 1 \text{ integer}}$, where

$$A_{x,L} = \left\{ \sigma \in \{0, 1\}^E; B(x, L) \text{ is linked to } \partial_{\text{int}} B(x, 2L) \right. \\ \left. \text{by a path where } \sigma \text{ vanishes} \right\}. \quad (3.5)$$

In particular, we see that in the notation of (0.10) and (1.36), for $u \geq 0$,

$$A_{x,L}^u \stackrel{(1.36)}{=} A_{x,L}(\mathcal{I}^u) = \{B(x, L) \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B(x, 2L)\}. \quad (3.6)$$

To describe the second example, we consider the family of “half-planes” \mathcal{P} in E of the form

$$\mathcal{P} = \{y(n); n \geq 0\} \times \mathbb{Z} \subseteq E, \quad (3.7)$$

where $y(n)$, $n \geq 0$, is a semi-infinite geodesic in G (recall $E = G \times \mathbb{Z}$), that is:

$$d_G(y(n), y(m)) = |n - m|, \quad \text{for all } n, m \geq 0. \quad (3.8)$$

It is well-known, see for instance Theorem 3.1 of [29], that for each $y \in G$, one can find such a $y(n)$, $n \geq 0$, with $y(0) = y$. Given such a half-plane \mathcal{P} we say that a finite sequence x_0, \dots, x_N in \mathcal{P} is a $*$ -path, when for each $0 \leq i < N$, $x_i \neq x_{i+1}$, and the G -projections of x_i and x_{i+1} lie at d_G -distance at most 1, and a similar condition holds for the \mathbb{Z} -projections.

We then define the family of events depending on the additional parameter \mathcal{P} varying over all possible “half-planes” in E , $\mathcal{B} = (B_{x,L,\mathcal{P}})$, where:

$$\begin{aligned} B_{x,L,\mathcal{P}} &= \left\{ \sigma \in \{0, 1\}^E; B(x, L) \text{ is linked to } \partial_{\text{int}} B(x, 2L) \text{ by a } * \text{-path in } \mathcal{P}, \right. \\ &\quad \left. \text{where } \sigma \text{ takes the value } 1 \right\}, \quad \text{when } x \in \mathcal{P}, \\ &= \phi, \quad \text{when } x \notin \mathcal{P}. \end{aligned} \quad (3.9)$$

In particular when $x \in \mathcal{P}$, $B_{x,L,\mathcal{P}}^u$ coincides with the event that appears in (0.13).

Proposition 3.2

\mathcal{A} is a family of decreasing events on $\{0, 1\}^E$ that cascades with complexity at most $\alpha + \frac{\beta}{2}$. (3.10)

\mathcal{B} is a family of increasing events on $\{0, 1\}^E$ that cascades uniformly with complexity at most $\frac{\beta}{2}$. (3.11)

Proof We begin with the proof of (3.10). The events $A_{x,L}$ are clearly decreasing and $\sigma(\Psi_{x'}, x' \in B(x, 10L))$ -measurable. Then note that given $x \in E$, $L \geq 1$, and ℓ multiple of 100, one can find x_1^i , $1 \leq i \leq N_1$, in $\partial_{\text{int}} B(x, \ell L)$ and x_2^j , $1 \leq j \leq N_2$, in $\partial_{\text{int}} B(x, \frac{3}{2} \ell N)$, such that

$$N_1 \vee N_2 \leq c \ell^{\alpha + \frac{\beta}{2}}, \quad (3.12)$$

$$\begin{aligned} \partial_{\text{int}} B(x, \ell L) &\subseteq \bigcup_{1 \leq i \leq N_1} B(x_1^i, L), \quad \text{and} \\ \partial_{\text{int}} B\left(x, \frac{3}{2} \ell L\right) &\subseteq \bigcup_{1 \leq j \leq N_2} B(x_2^j, L). \end{aligned} \quad (3.13)$$

Indeed we first select a maximal collection in $\partial_{\text{int}} B(x, \ell L)$ with mutual distance bigger than L , x_1^i , $1 \leq i \leq N_1$. The balls $B(x_1^i, \frac{L}{2})$ are thus pairwise disjoint, and each have, by (1.11), volume at least $c L^{\alpha + \frac{\beta}{2}}$. Their union is contained in $B(x, 2\ell L)$ and thus has volume at most $c \ell^{\alpha + \frac{\beta}{2}} L^{\alpha + \frac{\beta}{2}}$, using (1.11) once again. As a result $N_1 \leq c \ell^{\alpha + \frac{\beta}{2}}$. In a similar fashion we can construct x_2^j ,

$1 \leq j \leq N_2$, so that (3.12), (3.13) holds. Note that by construction one has

$$d(x_1^i, x_2^j) \geq \frac{\ell}{2} L - 1 > \frac{\ell}{5} L, \quad \text{for } 1 \leq i \leq N_1, 1 \leq j \leq N_2, \quad (3.14)$$

and defining Λ as the collection of points x_1^i and x_2^j , we see that (3.2), (3.3), hold with $\lambda = \alpha + \frac{\beta}{2}$.

Finally observe that any path from $B(x, \ell L)$ to $\partial_{\text{int}} B(x, 2\ell L)$ must visit $\partial_{\text{int}} B(x, \ell L)$ and thus enter one of the $B(x_1^i, L)$, then leave $B(x_1^i, 2L)$, then visit $\partial_{\text{int}} B(x, \frac{3}{2}\ell L)$ and thus enter one of the $B(x_2^j, L)$, and then leave $B(x_2^j, 2L)$ before reaching $\partial_{\text{int}} B(x, 2\ell L)$. This shows that

$$A_{x_*, \ell L} \subseteq \bigcup_{\substack{1 \leq i \leq N_1 \\ 1 \leq j \leq N_2}} A_{x_1^i, L} \cap A_{x_2^j, L}, \quad (3.15)$$

and due to (3.14) thus completes the proof of (3.10).

Let us now turn to the proof (3.11). The events $B_{x, L, \mathcal{P}}$ are clearly increasing and $\sigma(\Psi_{x'}, x' \in B(x, 10L))$ -measurable. When $x \in \mathcal{P}$, $R \geq 1$, then $B(x, R) \cap \mathcal{P}$ is a rectangle with horizontal sides of length comparable to R up to a multiplicative constant, and vertical sides of height comparable to $R^{\frac{\beta}{2}}$ up to a multiplicative constant. The set $\partial_{\text{int}}^{\mathcal{P}} B(x, R)$ of vertices of $B(x, R) \cap \mathcal{P}$ that neighbour $\mathcal{P} \setminus B(x, R)$ is the union of two horizontal and one or two vertical sides of the rectangle $B(x, R) \cap \mathcal{P}$. We then proceed as in the proof of (3.10), in essence replacing $\partial_{\text{int}} B(x, \ell L)$ and $\partial_{\text{int}} B(x, \frac{3}{2}\ell L)$ by $\partial_{\text{int}}^{\mathcal{P}} B(x, \ell L)$ and $\partial_{\text{int}}^{\mathcal{P}} B(x, \frac{3}{2}\ell L)$. In a similar fashion we find x_1^i , $1 \leq i \leq N_1$, in $\partial_{\text{int}}^{\mathcal{P}} B(x, \ell L)$ and x_2^j , $1 \leq j \leq N_2$, in $\partial_{\text{int}}^{\mathcal{P}} B(x, \frac{3}{2}\ell L)$, such that

$$N_1 \vee N_2 \leq c \ell^{\frac{\beta}{2}}, \quad (3.16)$$

$$\partial_{\text{int}}^{\mathcal{P}} B(x, \ell L) \subseteq \bigcup_{1 \leq i \leq N_1} B(x_1^i, L), \quad \partial_{\text{int}}^{\mathcal{P}} B\left(x, \frac{3\ell}{2} L\right) \subseteq \bigcup_{1 \leq j \leq N_2} B(x_2^j, L). \quad (3.17)$$

On the other hand when $x \notin \mathcal{P}$, (and $B_{x, \ell L, \mathcal{P}} = \emptyset$), we simply choose $N_1 = N_2 = 1$, and $x_1^i \in \partial_{\text{int}} B(x, \ell L)$, $x_2^j \in \partial_{\text{int}} B(x, \frac{3}{2}\ell L)$. Similar arguments as for the proof of (3.10) show that (3.11) holds. \square

Remark 3.3

- (1) In the case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, $\alpha + \frac{\beta}{2} = d + 1$ in (3.10). But one easily sees that in fact \mathcal{A} cascades with complexity at most d . This fact was for instance implicitly used in the proof of Lemma 1.2 of [19].

- (2) In the case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, Teixeira introduced in [27] a very interesting family of events having the cascading property, which we briefly describe below. We now use the sup-norm distance $d_\infty(\cdot, \cdot)$ in place of $d(\cdot, \cdot)$. For $x \in \mathbb{Z}^{d+1}$, and $L \geq 1$ integer, we define the separation event:

$$\begin{aligned} \tilde{S}_{x,L} = \{ \sigma \in \{0, 1\}^{\mathbb{Z}^{d+1}}; \text{ there exist two connected sets} \\ A_1, A_2 \subset x + [-L, 2L]^{d+1}, \text{ with diameter at least } \frac{L}{2}, \\ \text{separated by } \sigma \text{ in } x + [-2L, 3L]^{d+1} \}, \end{aligned} \quad (3.18)$$

where the expression “ σ separates A_1, A_2 in $x + [-2L, 3L]^{d+1}$ ” means that the mutual graph-distance between A_1 , and A_2 exceeds 1, i.e. $d_{\mathbb{Z}^{d+1}}(A_1, A_2) > 1$, and that any path in $x + [-2L, 3L]^{d+1}$ from ∂A_1 to ∂A_2 meets the set $\Sigma(\sigma) = \{x \in \mathbb{Z}^{d+1}, \sigma(x) = 1\}$.

In other words on the complement of the separation event $\tilde{S}_{x,L}$, for any two connected subsets A_1, A_2 of $x + [-L, 2L]^{d+1}$ with diameter at least $L/2$, and mutual graph-distance bigger than 1, one can find a path from ∂A_1 to ∂A_2 in $x + [-2L, 3L]^{d+1}$, where σ identically vanishes.

It then follows from Theorem 5.2 of [27] that

$$\tilde{S} \stackrel{\text{def}}{=} (\tilde{S}_{x,L})_{x \in \mathbb{Z}^{d+1}, L \geq 1} \text{ cascades with complexity at most } d + 1. \quad (3.19)$$

- (3) In analogy with [27], we introduce in the general context of the present work the collection $\mathcal{S} = (S_{x,L})_{x \in E, L \geq 1 \text{ integer}}$ of separation events defined by

$$\begin{aligned} S_{x,L} = \{ \sigma \in \{0, 1\}^E; \text{ there exist connected subsets } A_1 \text{ and } A_2 \\ \text{of } \overline{B(x, 3L)} \text{ with } d(\cdot, \cdot)\text{-diameter at least } L, \text{ separated} \\ \text{by } \Sigma(\sigma) \text{ in } B(x, 5L) \}, \end{aligned} \quad (3.20)$$

where $\Sigma(\sigma) = \{x \in E; \sigma(x) = 1\}$, and the above separation statement means that the mutual graph-distance $d_E(A_1, A_2)$ is bigger than 1, and that any path from ∂A_1 to ∂A_2 in $B(x, 5L)$ meets $\Sigma(\sigma)$.

We show in Proposition A.2 of the Appendix that

$$\begin{aligned} \mathcal{S} \text{ is a family of increasing events on } \{0, 1\}^E \text{ that cascades with} \\ \text{complexity at most } \alpha + \frac{\beta}{2}. \end{aligned} \quad (3.21)$$

As an aside, let us mention that making further progress on the question whether $u_* > 0$, when $\nu < 1$, might possibly involve the application of Corollary 3.7 below, to a suitable variation on the above family \mathcal{S} , see also Remark 5.6 (2).

We now come to the main result of this section, which combines the decoupling inequalities of Sect. 2 with the above notion of cascading property. We recall the sequence of length scales introduced in (2.1), as well as the notation (1.36), and (2.70). We implicitly assume that $\ell_0 \geq 10^6 c_0$, see (2.1), is divisible by 100, and that $L_0 \geq 1$ is an integer.

Theorem 3.4 ($K > 0$, $0 < v' < v$, $\ell_0 \geq c(K, v')$, $L_0 \geq 1$, $\lambda > 0$)

Consider $\mathcal{G} = (G_{x,L})_{x \in E, L \geq 1 \text{ integer}}$, a collection of decreasing, resp. increasing events on $\{0, 1\}^E$, cascading with complexity at most λ . Then for any $n \geq 0$, $u_0 > 0$, one has for decreasing events

$$\sup_{x \in E} \mathbb{P}[G_{x,L_n}^{u_0+}] \leq (c(\mathcal{G}, \lambda) \ell_0^{2\lambda})^{2^n-1} \left(\sup_{x \in E} \mathbb{P}[G_{x,L_0}^{u_0}] + \varepsilon(u_0) \right)^{2^n}, \quad (3.22)$$

resp. for increasing events

$$\sup_{x \in E} \mathbb{P}[G_{x,L_n}^{u_0-}] \leq (c(\mathcal{G}, \lambda) \ell_0^{2\lambda})^{2^n-1} \left(\sup_{x \in E} \mathbb{P}[G_{x,L_0}^{u_0}] + \varepsilon(u_0^-) \right)^{2^n}, \quad (3.23)$$

with $\varepsilon(\cdot)$ as in (2.73).

When the family of events depends on a parameter and cascades uniformly with complexity at most λ , similar inequalities hold as in (3.22), (3.23), where the supremum appearing on both sides of the inequalities now runs over $x \in E$ and the parameter set.

Proof It follows from (3.4), and induction on n that

$$G_{x,L_n}^u \subseteq \bigcup_{\mathcal{T} \in \Lambda_n^{\mathcal{G}}} \bigcap_{m \in \mathcal{T}_{(n)}} G_{x_m, \mathcal{T}, L_0}^u, \quad (3.24)$$

where the subset $\Lambda_n^{\mathcal{G}}$ of the collection Λ_n of embeddings of T_n , see above (2.2), has cardinality at most

$$\begin{aligned} |\Lambda_n^{\mathcal{G}}| &\leq (c(\mathcal{G}, \lambda) \ell_0^\lambda)^2 (c(\mathcal{G}, \lambda) \ell_0^\lambda)^4 \dots (c(\mathcal{G}, \lambda) \ell_0^\lambda)^{2^n} \\ &= (c'(\mathcal{G}, \lambda) \ell_0^{2\lambda})^{2^n-1}. \end{aligned} \quad (3.25)$$

In view of the decoupling inequalities (2.71), (2.72), our claims (3.22), (3.23) follow. The extension to families depending on a parameter and having the uniform cascading property is immediate. \square

We now derive two corollaries, which we will later apply in Sects. 4 and 5.

Corollary 3.5 Consider a family $\mathcal{G} = (G_{x,L})_{x \in E, L \geq 1 \text{ integer}}$ of events on $\{0, 1\}^E$ cascading with complexity at most $\lambda > 0$, and $u > 0$ such that

$$\lim_{L \rightarrow \infty} \sup_{x \in E} \mathbb{P}[G_{x,L}^u] = 0. \quad (3.26)$$

If the events in \mathcal{G} are decreasing, resp. increasing, then for $\bar{u} > u$, resp. $\bar{u} < u$, one can find integers $L_0 \geq 1$, $\ell_0 > 1$, such that with $L_n = \ell_0^n L_0$,

$$\sup_{x \in E} \mathbb{P}[G_{x,L_n}^{\bar{u}}] \leq 2^{-2^n}, \quad \text{for all } n \geq 0. \quad (3.27)$$

When the family depends on a parameter and cascades uniformly with complexity at most λ , if (3.26) holds with a joint supremum over x in E and the parameter set, then (3.27) holds with a similar modification.

Proof We pick $K = 2$ and $v' = \frac{v}{2}$ in Theorem 3.4, and from now on assume that $\ell_0 > c = c(K = 2, v' = \frac{v}{2}) \geq 10^6 c_0$ is a multiple of 100 such that Theorem 3.4 applies. Setting $u_0 = u$, we further assume that $\ell_0 \geq c(\bar{u}, \mathcal{G}, \lambda)$ is large enough so that in the case of decreasing events, (cf. (2.70), (3.22) for the notation):

$$\begin{aligned} \text{(i)} \quad & u_\infty^+ < \bar{u}, \quad \text{and} \\ \text{(ii)} \quad & c(\mathcal{G}, \lambda) \ell_0^{2\lambda} \varepsilon(u) \stackrel{(2.73)}{\leq} c(\mathcal{G}, \lambda) \ell_0^{2\lambda} 2e^{-2u\ell_0^{v'}} / (1 - e^{-2u\ell_0^{v'}}) \leq \frac{1}{4}, \end{aligned} \quad (3.28)$$

for all $L_0 \geq 1$,

and that in the case of increasing events:

$$\begin{aligned} \text{(i)} \quad & u_\infty^- > \max\left(\bar{u}, \frac{u}{2}\right) \\ \text{(ii)} \quad & c(\mathcal{G}, \lambda) \ell_0^{2\lambda} \varepsilon(u_\infty^-) \leq c(\mathcal{G}, \lambda) \ell_0^{2\lambda} \varepsilon\left(\frac{u}{2}\right) \leq \frac{1}{4}, \quad \text{for all } L_0 \geq 1. \end{aligned} \quad (3.29)$$

So in the case of decreasing events we see that for all $L_0 \geq 1$, $n \geq 0$, $x \in E$,

$$\mathbb{P}[G_{x,L_n}^{\bar{u}}] \leq \mathbb{P}[G_{x,L_n}^{u_\infty^+}] \stackrel{(3.22)}{\leq} \stackrel{(3.28) \text{ (ii)}}{\leq} \left[c(\mathcal{G}, \lambda) \ell_0^{2\lambda} \sup_{x \in E} \mathbb{P}[G_{x,L_0}^u] + \frac{1}{4} \right]^{2^n}. \quad (3.30)$$

In view of (3.26) we can pick a large enough L_0 so that (3.27) holds. When the family consists of increasing events, the same argument with (3.23) and (3.29) in place of (3.22), (3.23) yields the claim. The case of a family depending on a parameter is handled in a similar fashion as above. \square

Remark 3.6

- (1) Note that we can pick ℓ_0 and L_0 in Corollary 3.5 as $c(\bar{u}, \mathcal{G})$, where the convention concerning constants can be found at the end of the Introduction. For instance we can select $\ell_0 > 1$ minimal such that (3.27) holds for some $L_0 \geq 1$ and then given this choice of ℓ_0 pick a minimal L_0 such that (3.27) holds.
- (2) The above corollary could in fact accommodate the choice of a faster growth of complexity than what is imposed on cascading families in (3.3). One could as well have chosen $c(\mathcal{G}, \tilde{\nu}) e^{\ell \tilde{\nu}}$, with $\tilde{\nu} < \nu$, in place of $c(\mathcal{G}, \lambda) \ell^\lambda$, and the above proof would have gone through with minor modifications. Apart from the various examples presented in Proposition 3.2 and Remark 3.3, one motivation for the choice that appears in (3.3) stems from the next proposition, where the parameter λ from (3.3) plays an explicit role.

We are now ready to state the second corollary of Theorem 3.4, which will be used in Sect. 5. The following observation will be implicit in the interpretation of the expression in (3.31) below: given X_1^1, \dots, X_1^M independent canonical random walks on E , the indicator function $\chi_{\cup_1^M \text{range}(X_i^1)}$ of the union of the ranges of the walks defines a $\{0, 1\}^E$ -valued random variable (when $M = 0$ this random variable is constant and equal to the function identically equal to 0 on E).

Corollary 3.7 *Consider a family $\mathcal{G} = (G_{x,L})_{x \in E, L \geq 1 \text{ integer}}$ of increasing events on $\{0, 1\}^E$ cascading with complexity at most $\lambda > 0$, and an integer $M \geq \lfloor \frac{2\lambda}{\nu} \rfloor$ (recall $\nu = \alpha - \frac{\beta}{2} > 0$). Assume that*

$$\lim_{L \rightarrow \infty} \sup_{x, x_1, \dots, x_M} \bigotimes_{i=1}^M P_{x_i} [\chi_{\cup_1^M \text{range}(X_i^1)} \in G_{x,L}] = 0, \quad (3.31)$$

where the supremum runs over $x \in E, x_1, \dots, x_M \in \partial_{\text{int}} B(x, 20L)$. Then one can find $u > 0$, and integers $L_0 \geq 1, \ell_0 > 1$, so that

$$\sup_{x \in E} \mathbb{P}[G_{x, L_n}^u] \leq 2^{-2n}, \quad \text{for all } n \geq 0. \quad (3.32)$$

In addition when the family depends on a parameter and cascades uniformly with complexity at most λ , if (3.31) holds with a joint supremum over x, x_1, \dots, x_M as above and the parameter set, then the supremum in (3.32) can be replaced by a joint supremum over x and the parameter set.

Proof We choose $K = 2$ and $0 < \nu'' < \nu' < \nu = \alpha - \frac{\beta}{2}$, so that

$$\nu''(M+1) > 2\lambda. \quad (3.33)$$

We know from Theorem 3.4 that for $\ell_0 \geq c(K = 2, v')$ multiple of 100, $u_0 > 0$, $L_0 \geq 1$, one has

$$\sup_{x \in E} \mathbb{P}[G_{x, L_n}^{u_\infty^-}] \leq \left[c(\mathcal{G}, \lambda) \ell_0^{2\lambda} \left(\sup_{x \in E} \mathbb{P}[G_{x, L_0}^{u_0}] + \varepsilon(u_\infty^-) \right) \right]^{2^n}. \quad (3.34)$$

From now on we assume ℓ_0 large enough so that, cf. (2.70), for all $u_0 > 0$,

$$u_\infty^- \geq \frac{1}{2} u_0, \quad (3.35)$$

and such that setting

$$u_0 = L_0^{-v} \ell_0^{-v''}, \quad (3.36)$$

one also has, cf. (2.73):

$$c(\mathcal{G}, \lambda) \ell_0^{2\lambda} \varepsilon(u_\infty^-) \leq c(\mathcal{G}, \lambda) \ell_0^{2\lambda} 2e^{-\ell_0^{v'-v''}} / (1 - e^{-\ell_0^{v'-v''}}) \leq \frac{1}{4},$$

for all $L_0 \geq 1$. (3.37)

The claim (3.32) will then follow in view of (3.34), (3.37), once we show that

$$\lim_{\ell_0 \rightarrow \infty} \lim_{L_0 \rightarrow \infty} \ell_0^{2\lambda} \sup_{x \in E} \mathbb{P}[G_{x, L_0}^{u_0}] = 0, \quad (3.38)$$

with u_0 as in (3.36) and ℓ_0 multiple of 100.

For this purpose we observe that G_{x, L_0} is $\sigma(\Psi_{x'}, x' \in B(x, 10L_0))$ -measurable and $\mathcal{I}^{u_0} \cap B(x, 10L_0)$ coincides by (1.32) with the trace on $B(x, 10L_0)$ of the union of the ranges of the trajectories in the support of the Poisson point measure $\mu \stackrel{\text{def}}{=} \mu_{B(x, 20L_0), u_0}$. The intensity of μ equals $u_0 P_{e_{B(x, 20L_0)}}$ by (1.27), and the law of μ can thus be generated as the sum of a Poisson number of point masses on W^+ located at independent random walk trajectories with common distribution $P_{\bar{e}}$, where the common starting distribution equals $\bar{e} = e_{B(x, 20L_0)}/\text{cap}(B(x, 20L_0))$ and the Poisson variable has intensity

$$u_0 \text{cap}(B(x, 20L_0)) \stackrel{(1.26)}{\leq} c u_0 L_0^v \stackrel{(3.36)}{\leq} c \ell_0^{-v''} \stackrel{\text{def}}{=} \kappa. \quad (3.39)$$

The probability that the Poisson variable exceeds M is at most:

$$e^{-\kappa} \sum_{k \geq M+1} \frac{\kappa^k}{k!} \leq \kappa^{M+1} = c^{(M+1)} \ell_0^{-v''(M+1)}.$$

As a result we see that

$$\begin{aligned} & \ell_0^{2\lambda} \sup_{x \in E} \mathbb{P}[G_{x, L_0}^{u_0}] \\ & \leq c^{(M+1)} \ell_0^{2\lambda - (M+1)v''} \\ & \quad + \ell_0^{2\lambda} \sup_{x, x_1, \dots, x_M} \bigotimes_{i=1}^M P_{x_i} [X_{\cup_1^M \text{range}(X^i)} \in G_{x, L_0}^{u_0}], \end{aligned} \quad (3.40)$$

where the supremum runs over the same collection as in (3.31). By (3.31) and (3.33) we see that (3.38) holds. This proves (3.32). The case of a family \mathcal{G} depending on a parameter is handled in a similar fashion. This concludes the proof of Corollary 3.7. \square

Remark 3.8

- (1) In the case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, the family $\tilde{\mathcal{S}}$ of separation events, cf. (3.18), (3.19) introduced by Teixeira [27] cascades with complexity at most $d+1$, and $v = d-1$, so that $\lfloor \frac{2\lambda}{v} \rfloor = 2 + \lfloor \frac{4}{d-1} \rfloor$. When $d \geq 4$, we can choose $M = 3$, and it follows from Theorem 6.11 of [27] that (3.31) holds. As a result we see by (3.32) that one can find $u > 0$, $\ell_0 > 1$, and $L_0 \geq 1$, such that

$$\mathbb{P}[\tilde{\mathcal{S}}_{0, L_n}^u] \leq 2^{-2^n}, \quad \text{for all } n \geq 0, \quad (3.41)$$

(the supremum over x has been dropped due to translation invariance).

- (2) In view of the above remark, the family \mathcal{S} in (3.20) is also a natural candidate for which condition (3.31) of Corollary 3.7 ought to be tested. We hope this point will be tackled elsewhere.
- (3) By similar considerations as in Remark 3.6, we can pick u , ℓ_0 , L_0 as $c(\mathcal{G})$ in the above Corollary 3.7.

4 Finiteness of u_* and connectivity bounds

In this section we explore the percolative properties of the vacant set of random interlacements on E , in the sub-critical phase of the model. We show that in our set-up the critical parameter u_* in (0.7) is always finite, and we derive stretched exponential bounds on the connectivity function $\mathbb{P}[x \overset{y^u}{\longleftrightarrow} y]$, when $u > u_{**}$, with $u_{**} \geq u_*$, a certain finite critical value, possibly equal to u_* , see (0.10). The main result appears in Theorem 4.1, and comes as an application of Theorem 3.4 and Corollary 3.5. Even in the case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, Theorem 4.1 improves on what is presently known, see Remark 4.2 (1) below.

We recall the definition of u_{**} from (0.10),

$$u_{**} = \inf \left\{ u \geq 0; \lim_{L \rightarrow \infty} \sup_{x \in E} \mathbb{P}[B(x, L) \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B(x, 2L)] = 0 \right\} \\ \in [0, \infty]. \quad (4.1)$$

With the notation from (0.3), it is plain that $\eta(x, u) = 0$ for all x in E , when $u > u_{**}$, and therefore we see that

$$0 \leq u_* \leq u_{**} \leq \infty. \quad (4.2)$$

A routine covering argument further shows that u_{**} does not change when we restrict L to integer values in (4.1), see also below (4.5). The following theorem will in particular show that u_{**} is finite.

Theorem 4.1

$$0 \leq u_* \leq u_{**} < \infty, \quad (4.3)$$

and for $u > u_{**}$ there exist $c_4(u), c_5(u) > 0, 0 < \kappa(u) < 1$, such that

$$\sup_{x \in E} \mathbb{P}[B(x, L) \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B(x, 2L)] \leq c_4 e^{-c_5 L^\kappa}, \quad \text{for } L \geq 1. \quad (4.4)$$

Proof Once we know that $u_{**} < \infty$, it follows from Corollary 3.5 applied to the collection \mathcal{A} of (3.5), see also Remark 3.6, that for $u > u_{**}$, there is $\ell_0(u) > 1$ and $L_0(u) \geq 1$, such that:

$$\sup_{x \in E} \mathbb{P}[A_{x, L_n}^u] \leq 2^{-2^n}, \quad \text{for all } n \geq 0. \quad (4.5)$$

If we now consider $L \geq 4L_0$, and $L_n \leq L/4 < L_{n+1}$, a similar argument as in the proof of Proposition 3.2 enables us to cover $B(x, L)$ by a collection of at most $c(u)$ closed balls of radius L_n with centers in $B(x, L)$. It then follows that

$$\begin{aligned} & \mathbb{P}[B(x, L) \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B(x, 2L)] \\ & \leq c(u) \sup_{x \in E} \mathbb{P}[A_{x, L_n}^u] \\ & \leq c(u) 2^{-2^n} = c(u) \exp \left\{ - \left(\frac{L_{n+1}}{L_1} \right)^\kappa \right\} \\ & \leq c(u) \exp \{ - L_1^{-\kappa} L^\kappa \}, \quad \text{with } \kappa(u) = \frac{\log 2}{\log \ell_0}, \end{aligned} \quad (4.6)$$

and (4.4) follows straightforwardly.

We hence only need to show that u_{**} is finite. For this purpose we use (3.22) with $\mathcal{G} = \mathcal{A}$, $\lambda = \alpha + \frac{\beta}{2}$, $K = 1$, $v' = \frac{v}{2}$, and find that for some $\ell_0 > 1$, $L_0 = 1$, and all $n \geq 0$, $u_0 > 0$, one has

$$\begin{aligned} & \sup_{x \in E} \mathbb{P}[B(x, L_n) \xleftrightarrow{\mathcal{V}_\infty^+} \partial_{\text{int}} B(x, 2L_n)] \\ & \leq \left[c \ell_0^{2\lambda} \left(\sup_{x \in E} \mathbb{P}[B(x, 1) \xleftrightarrow{\mathcal{V}^{u_0}} \partial_{\text{int}} B(x, 2)] + \varepsilon(u_0) \right) \right]^{2^n}. \end{aligned} \quad (4.7)$$

Taking into account that the event under the probability in the right-hand side of (4.7) is contained in the event $\{\mathcal{V}^{u_0} \cap B(x, 2) \neq \emptyset\}$, we see that

$$\sup_{x \in E} \mathbb{P}[B(x, 1) \xleftrightarrow{\mathcal{V}^{u_0}} \partial_{\text{int}} B(x, 2)] \stackrel{(1.37)}{\leq} c e^{-cu_0}. \quad (4.8)$$

Thus for large enough u_0 , the left-hand side of (4.7) is at most 2^{-2^n} for all $n \geq 0$. This proves that $u_{**} < \infty$, and concludes the proof of Theorem 4.1. \square

Remark 4.2

- (1) In the case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, when we use the sup-norm distance in place of $d(\cdot, \cdot)$, the above result improves upon what is known from [19]. Indeed it shows that

$$\begin{aligned} u_{**} &= \inf\{u \geq 0; \lim_{L \rightarrow \infty} \mathbb{P}[B_\infty(0, L) \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B_\infty(0, 2L)] = 0\} \\ &= \inf\{u \geq 0; \text{for some } \alpha > 0, \\ & \quad \lim_{L \rightarrow \infty} L^\alpha \mathbb{P}[B_\infty(0, 2L) \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B_\infty(0, 2L)] = 0\}, \end{aligned} \quad (4.9)$$

the second line being a priori bigger or equal to u_{**} , but in fact equal to u_{**} thanks to (4.4). The quantity in the second line of (4.9) is used for the definition of u_{**} in [19], and the equality stated in (4.9) is new.

- (2) Can one construct examples where $u_{**} > u_*$ holds? Can this bring some light concerning the open question whether $u_* = u_{**}$ in the case of $E = \mathbb{Z}^{d+1}$, $d \geq 2$? The absence of a true substitute for the BK Inequality in the context of interlacement percolation makes this last question hard to answer at present.
- (3) Note that for any two distinct points x, x' in E , with $d(x, x') = L + 1$, one has the inclusion

$$\{x \xleftrightarrow{\mathcal{V}^u} x'\} \subseteq \{x \xleftrightarrow{\mathcal{V}^u} \partial_{\text{int}} B(x, L)\},$$

because any path in E from x to x' must at some point exit $B(x, L)$. This observation combined with (4.4) implies that for $u > u_{**}$ the connectivity function has a stretched exponential decay:

$$\mathbb{P}[x \overset{\mathcal{V}^u}{\longleftrightarrow} x'] \leq c(u) e^{-c'(u)d(x,x')^\kappa} \quad \text{for } x, x' \text{ in } E, \quad (4.10)$$

with κ as in (4.4).

- (4) It is plain that the argument employed in the proof of Theorem 4.1 is quite robust and could be adapted to different collections of decreasing events having the cascading property.

5 Positivity of u_* and connectivity in half-planes

In this section we explore the percolative properties of the vacant set of random interlacements on E in the super-critical phase of the model. We show that $u_* > 0$ when $\alpha \geq 1 + \frac{\beta}{2}$, i.e. when $\nu \geq 1$, and that for small $u > 0$, \mathcal{V}^u percolates in half-planes, see (3.7) for the definition of half-planes. Additionally we derive stretched exponential bounds on the $*$ -connectivity function of \mathcal{I}^u inside half-planes. Our main results appear in Theorem 5.1 and Corollary 5.5. The proof of Theorem 5.1 has some flavor of the proof of Theorem 5.3 of [20], where a similar restriction $\alpha \geq 1 + \frac{\beta}{2}$ is present. This restriction rules out examples such as $E = G \times \mathbb{Z}$, where G is the discrete skeleton of the Sierpinski gasket, cf. [3, 15]. It is an interesting question whether $u_* > 0$ remains true in this case. Let us point out that when $E = \mathbb{Z}^{d+1}$, $d \geq 2$, the condition $\nu \geq 1$ is automatically fulfilled and the results of this section go beyond present knowledge, cf. [19, 26, 27].

We recall the definition of half-planes in (3.7) and of $*$ -paths below (3.8). We then introduce

$$\tilde{u} = \inf \left\{ u \geq 0; \lim_{L \rightarrow \infty} \sup_{\mathcal{P}, x \in \mathcal{P}} \mathbb{P}[B(x, L) \overset{\mathcal{I}^u \cap \mathcal{P}}{\longleftrightarrow} \partial_{\text{int}} B(x, 2L)] > 0 \right\}, \quad (5.1)$$

(when L is a positive integer, the event inside the probability coincides with $B_{x,L,\mathcal{P}}^u$, where $B_{x,L,\mathcal{P}}$ has been defined in (3.9)). The main result of this section is the following.

Theorem 5.1

$$0 \leq \tilde{u} \leq u_*, \quad (5.2)$$

moreover for $u < \tilde{u}$,

$$\mathbb{P}\text{-a.s.}, \mathcal{V}^u \text{ percolates in every half-plane } \mathcal{P}, \quad (5.3)$$

and there exist positive $c_6(u)$, $c_7(u)$, and $0 < \tilde{\kappa}(u) < 1$, such that

$$\sup_{\mathcal{P}, x \in \mathcal{P}} \mathbb{P}[B(x, L) \overset{*}{\longleftrightarrow}^{\mathcal{I}^u \cap \mathcal{P}} \partial_{\text{int}} B(x, 2L)] \leq c_6 e^{-c_7 L^{\tilde{\kappa}}}, \quad \text{for } L \geq 1. \quad (5.4)$$

In addition when $\alpha \geq 1 + \frac{\beta}{2}$, i.e. $\nu \geq 1$, one has

$$0 < \tilde{u} \leq u_*. \quad (5.5)$$

Proof The claim (5.4) is a direct application of Corollary 3.5, and similar considerations as below (4.5). The claim (5.2) follows immediately once we prove (5.3). We now prove (5.3). To this end we consider $u < \tilde{u}$ and $\mathcal{P} = \{y(n); n \geq 0\} \times \mathbb{Z}$, where $y(n)$, $n \geq 0$, is a semi-infinite geodesic in G . We use the notation $x_0 = (y(0), 0)$, and $\mathbb{L}_+ = \{y(0)\} \times [0, \infty)$, $\mathbb{L}_- = \{y(0)\} \times (-\infty, -1]$. The claim (5.3) is a consequence of

$$\mathbb{P}[x_0 \text{ belongs to an infinite connected component of } \mathcal{V}^u \cap \mathcal{P}] > 0. \quad (5.6)$$

Indeed a similar argument as in the proof of (1.39), see also (1.42), shows that a zero-one law holds for the probability that \mathcal{V}^u percolates in \mathcal{P} . Now for any integer $M > 0$, the probability in (5.6) is at least

$$\begin{aligned} & \mathbb{P}[B(x_0, M) \cap \mathcal{P} \subseteq \mathcal{V}^u \text{ and there is no } *- \text{path in } \mathcal{I}^u \cap \mathcal{P} \text{ from} \\ & \quad \mathbb{L}_+ \setminus B(x_0, M) \text{ to } \mathbb{L}_- \setminus B(x_0, M)] \\ & \stackrel{(1.34)}{\geq} \mathbb{P}[B(x_0, M) \cap \mathcal{P} \subseteq \mathcal{V}^u] \mathbb{P}[\text{there is no } *- \text{path in } \mathcal{I}^u \cap \mathcal{P} \text{ from} \\ & \quad \mathbb{L}_+ \setminus B(x_0, M) \text{ to } \mathbb{L}_- \setminus B(x_0, M)]. \end{aligned} \quad (5.7)$$

The first factor in the last line equals $\exp\{-u \text{cap}(B(x_0, M) \cap \mathcal{P})\} > 0$. As for the last factor, we consider the complement of the event under the probability, and keep track of the largest $B(x_0, 2^k M)$ not containing the rightmost point of the $*$ -path on $\{(y(n), 0); n \geq 0\}$. Setting $\tilde{x}_k = (y(2^{k+1}M + 1), 0)$, for $k \geq 0$, we see that the last factor is at least

$$\begin{aligned} & 1 - \sum_{k \geq 0} \mathbb{P}[B(\tilde{x}_k, 2^k M) \overset{*}{\longleftrightarrow}^{\mathcal{I}^u \cap \mathcal{P}} \partial_{\text{int}} B(\tilde{x}_k, 2^{k+1} M)] \\ & \stackrel{(5.4)}{\geq} 1 - \sum_{k \geq 0} c(u) e^{-c'(u)(2^k M)^{\tilde{\kappa}}} > 0, \quad \text{when } M \text{ is large.} \end{aligned}$$

This proves (5.6) and thus completes the proof of (5.3).

We now turn to the proof of (5.5) and assume from now on unless otherwise specified that $\nu = \alpha - \frac{\beta}{2} \geq 1$. In essence this assumption makes it hard for the

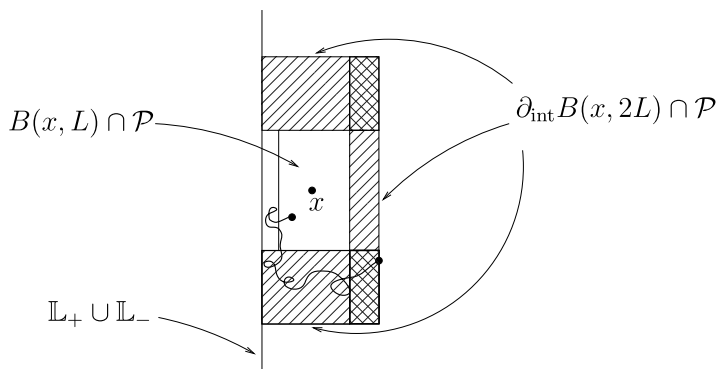


Fig. 2 An illustration with three rectangles bordering $\partial_{\text{int}} B(x, L) \cap \mathcal{P}$ and a $*$ -path in \mathcal{P} from $B(x, L) \cap \mathcal{P}$ to $\partial_{\text{int}} B(x, 2L) \cap \mathcal{P}$ inducing a crossing of one of the three rectangles

random walk on E to have a trace in a half-plane that covers a large $*$ -path. We define the integer M via

$$M = \left\lceil \frac{\beta}{\nu} \right\rceil. \quad (5.8)$$

In view of (3.11) and Corollary 3.7, (5.4) will hold for small $u > 0$, and our claim (5.5) will follow once we prove that:

$$\lim_{L \rightarrow \infty} \widetilde{\sup} \bigotimes_{i=1}^M P_{x_i} [\chi_{\cup_1^M \text{range}(X_i)} \in B_{x, L, \mathcal{P}}] = 0, \quad (5.9)$$

where the notation $\widetilde{\sup}$ refers to a supremum over \mathcal{P} , x in \mathcal{P} , x_1, \dots, x_M in $\partial_{\text{int}} B(x, 20L)$, and $B_{x, L, \mathcal{P}}$ appears in (3.9).

Given a half-plane \mathcal{P} and x in \mathcal{P} , we can find, depending on x and $L \geq 1$, three or four rectangles $\tilde{D} = \tilde{W} \times \tilde{J}$ in $\mathcal{P} \cap (\partial_{\text{int}} B(x, L) \cup (B(x, 2L) \setminus B(x, L)))$, bordering $\partial_{\text{int}} B(x, L) \cap \mathcal{P}$, with $L \leq |\tilde{W}| \leq 5L$, $[L^{\beta/2}] \leq |\tilde{J}| \leq [(5L)^{\beta/2}]$, and such that any $*$ -path in \mathcal{P} from $B(x, L) \cap \mathcal{P}$ to $\partial_{\text{int}} B(x, 2L) \cap \mathcal{P}$ contains a $*$ -path in \mathcal{P} joining the opposite sides of one of these rectangles (see Fig. 2). We now define

$$H = \left\lceil (L/N)^{\frac{\beta}{2}} \right\rceil, \text{ where } N = \log L, \text{ and } L \geq 10^3 \text{ is large enough} \\ \text{so that } 10^3 \leq H < \left\lceil L^{\frac{\beta}{2}} \right\rceil. \quad (5.10)$$

We can cover each of the three or four above rectangles \tilde{D} by at most $cN^{\beta/2}$ overlapping rectangles D having same horizontal projection as \tilde{D} and a vertical projection of cardinality H , so that every $*$ -path joining opposite sides of \tilde{D} has a trace in at least one of the rectangles D with either a horizontal projection containing a segment of at least $L/10$ points, or a vertical projection containing a segment of at least $H/10$ points. To see this, one for

instance considers the collection of rectangles D “vertical translates” of the form $\tilde{W} \times ([1, H] + k[H/100])$, $k \in \mathbb{Z}$, that intersect $\tilde{D} = \tilde{W} \times \tilde{J}$. One then looks at the location of the $*$ -path when it reaches the “middle” of the side of \tilde{D} it crosses, and picks one of the rectangles D with “closest center” to that location.

From the above considerations we see that for large L , the expression inside the \liminf in (5.9) is bounded from above by

$$A \stackrel{\text{def}}{=} c N^{\frac{\beta}{2}} \overline{\sup} \bigotimes_{i=1}^M P_{x_i} \left[\bigcup_{i=1}^M \text{range}(X^i) \cap D \text{ has } G\text{-projection with at least } \frac{L}{10} \text{ points or } \mathbb{Z}\text{-projection with at least } \frac{H}{10} \text{ points} \right], \quad (5.11)$$

where the notation $\overline{\sup}$ refers to a supremum over \mathcal{P} , x_1, \dots, x_M in \mathcal{P} , and $D = W \times J$, a rectangle in \mathcal{P} , with $L \leq |W| \leq 5L$, and $|J| = H$.

For a rectangle as above we introduce the notation

$$J_y = \{y\} \times J, \quad y \in G, \quad W_z = W \times \{z\}, \quad z \in \mathbb{Z}. \quad (5.12)$$

The following lemma comes as a preliminary step in bounding A in (5.11). It will enable us to control the average number of vertical segments J_y , $y \in W$, and horizontal segments W_z , $z \in J$, of D visited by a random walk on E starting from an arbitrary location. The bound on A will then follow by an application of Khashminskii’s lemma, see [16]. The calculations are similar to what appears in the proof of Corollary 5.3 of [20].

Lemma 5.2 ($v \geq 1$, $L \geq c$)

When $x = (y, z)$, $\bar{x} = (\bar{y}, \bar{z})$ are in D ,

$$P_x[H_{J_{\bar{y}}} < \infty] \leq \begin{cases} c(H \wedge d_G(y, \bar{y})^{\beta/2})/d_G(y, \bar{y})^v, & \text{when } \alpha > \beta, \\ c\{1 + \log(H^{\frac{2}{\beta}}/d_G(y, \bar{y}))\}/\log H, \\ \quad \text{when } \alpha = \beta \text{ and } 0 < d_G(y, \bar{y}) < H^{\frac{2}{\beta}}, \\ c H d_G(y, \bar{y})^{-\frac{\beta}{2}}/\log H, \\ \quad \text{when } \alpha = \beta \text{ and } d_G(y, \bar{y}) \geq H^{\frac{2}{\beta}}, \\ c(1 \wedge (H^{\frac{2}{\beta}}/d_G(y, \bar{y}))^v), & \text{when } \alpha < \beta, \end{cases} \quad (5.13)$$

as well as

$$P_x[H_{W_{\bar{z}}} < \infty] \leq \begin{cases} c(1 \wedge |z - \bar{z}|^{\frac{2}{\beta}(1-\nu)}), & \text{if } \nu > 1, \\ c(1 + \log(L/|z - \bar{z}|^{\frac{2}{\beta}}))/\log L, & \text{if } \nu = 1. \end{cases} \quad (5.14)$$

Proof We begin with the proof of (5.13). We apply the right-hand inequality of (1.25) with $K = J_{\bar{y}}$. With this in mind we introduce the function

$$\mathcal{J}_{\bar{y}}(x') = \sum_{\tilde{x} \in J_{\bar{y}}} g(x', \tilde{x}), \quad \text{for } x' \in E. \quad (5.15)$$

We note that by (1.9) when $x' \in J_{\bar{y}}$, one has

$$\mathcal{J}_{\bar{y}}(x') \geq c \sum_{\ell=1}^H \ell^{-(\frac{2\alpha}{\beta}-1)} \geq \begin{cases} c, & \text{when } \alpha > \beta, \\ c \log H, & \text{when } \alpha = \beta, \\ c H^{2-\frac{2\alpha}{\beta}}, & \text{when } \alpha < \beta. \end{cases} \quad (5.16)$$

On the other hand when $x \notin J_{\bar{y}}$, with x as in (5.13), it follows from (1.9) that

$$\mathcal{J}_{\bar{y}}(x) \leq c \sum_{\ell=1}^H (d_G(y, \bar{y})^\nu \vee \ell^{\frac{2\alpha}{\beta}-1})^{-1}, \quad (5.17)$$

so that when $d_G(y, \bar{y})^{\frac{\beta}{2}} \leq H$, keeping track of whether $\ell \leq d_G(y, \bar{y})^{\beta/2}$ or not, we find that

$$\mathcal{J}_{\bar{y}}(x) \leq \begin{cases} c d_G(y, \bar{y})^{\frac{\beta}{2}-\nu}, & \text{when } \alpha > \beta, \\ c(1 + \log(H/d_G(y, \bar{y})^{\frac{\beta}{2}})), & \text{when } \alpha = \beta, \\ c H^{2-\frac{2\alpha}{\beta}}, & \text{when } \alpha < \beta. \end{cases} \quad (5.18)$$

On the other hand when $d_G(y, \bar{y})^{\frac{\beta}{2}} > H$, we find instead in all cases that

$$\mathcal{J}_{\bar{y}}(x) \leq c H d_G(y, \bar{y})^{-\nu}. \quad (5.19)$$

The right-hand inequality in (1.25) yields that $P_x[H_{J_{\bar{y}}} < \infty]$ is bounded by $\mathcal{J}_{\bar{y}}(x)/\inf_{x' \in J_{\bar{y}}} \mathcal{J}_{\bar{y}}(x')$, and (5.13) now follows (note that $\nu = \frac{\beta}{2}$, when $\alpha = \beta$).

We then continue with the proof of (5.14), and now introduce the function

$$\mathcal{W}_{\bar{z}}(x') = \sum_{\tilde{x} \in W_{\bar{z}}} g(x', \tilde{x}), \quad \text{for } x' \in E. \quad (5.20)$$

We find by (1.9) that when $x' \in W_{\bar{z}}$, (remember that $L \leq |W| \leq 5L$), one has:

$$\begin{aligned} \mathcal{W}_{\bar{z}}(x') &\geq c \sum_{k=1}^{|W|} \ell^{-\nu} \geq c, \quad \text{when } \nu > 1, \\ &\geq c \log L, \quad \text{when } \nu = 1. \end{aligned} \quad (5.21)$$

On the other hand, when $x \notin W_{\bar{z}}$, with x as in (5.14), we have by (1.9):

$$\begin{aligned} \mathcal{W}_{\bar{z}}(x) &\leq c \sum_{\ell=1}^{|W|} (\ell^\nu \vee |z - \bar{z}|^{\frac{2\alpha}{\beta}-1})^{-1} \\ &\leq c |z - \bar{z}|^{\frac{2}{\beta}+1-\frac{2\alpha}{\beta}} + c \sum_{|z-\bar{z}|^{\frac{2}{\beta}} < \ell \leq |W|} \ell^{-\nu} \\ &\leq c |z - \bar{z}|^{\frac{2}{\beta}(1-\nu)}, \quad \text{when } \nu > 1, \\ &\leq c(1 + \log(L/|z - \bar{z}|^{\frac{2}{\beta}})), \quad \text{when } \nu = 1. \end{aligned} \quad (5.22)$$

Using once again the right-hand inequality of (1.25) we find that $P_x[H_{W_{\bar{z}}} < \infty]$ is bounded by $\mathcal{W}_{\bar{z}}(x)/\inf_{x' \in W_{\bar{z}}} \mathcal{W}_{\bar{z}}(x')$, and (5.14) follows. \square

With the help of the above lemma, we will now derive an upper bound on the expected number of vertical or horizontal segments in D visited by the random walk in E . We first introduce some notation. As a consequence of the strong Markov property at time H_D , we see that

$$N_{\text{vert}} \stackrel{\text{def}}{=} \sup_{x \in E} E_x \left[\sum_{\bar{y} \in W} 1\{H_{J_{\bar{y}}} < \infty\} \right] = \sup_{x \in D} E_x \left[\sum_{\bar{y} \in W} 1\{H_{J_{\bar{y}}} < \infty\} \right], \quad (5.23)$$

and that

$$N_{\text{hor}} \stackrel{\text{def}}{=} \sup_{x \in E} E_x \left[\sum_{\bar{z} \in J} 1\{H_{W_{\bar{z}}} < \infty\} \right] = \sup_{x \in D} E_x \left[\sum_{\bar{z} \in J} 1\{H_{W_{\bar{z}}} < \infty\} \right], \quad (5.24)$$

where $D \subseteq \mathcal{P}$ is a rectangle of the form $D = W \times J$, with $L \leq |W| \leq 5L$, and $|J| = H(< L^{\frac{\beta}{2}})$, see (5.10). The following lemma encapsulates the estimates that we will use, and that are stated in a simplified form to avoid tracking all special values of the parameters.

Lemma 5.3 ($\nu \geq 1, L \geq c$)

$$N_{\text{vert}} \leq c H^{\frac{2}{\beta}} (1 + \log(L/H^{\frac{2}{\beta}})) / (1 + 1\{\alpha = \beta\} \log L), \quad (5.25)$$

$$N_{\text{hor}} \leq c H(1 + \log(L/H^{\frac{2}{\beta}}))/\log L. \quad (5.26)$$

Proof We begin with the proof of (5.25). When $\alpha > \beta$, by the first line of (5.13) we find

$$\begin{aligned} N_{\text{vert}} &\leq \sum_{1 \leq \ell < H^{\frac{2}{\beta}}} c \ell^{-(\alpha-\beta)} + \sum_{H^{\frac{2}{\beta}} \leq \ell \leq |W|} c H \ell^{-\nu} \\ &\leq c H^{\frac{2}{\beta}(1+\beta-\alpha)+} + c(\log H) 1\{\alpha = \beta + 1\}, \end{aligned} \quad (5.27)$$

whence (5.25).

When $\alpha = \beta$, using the second and third line of (5.13) we find that:

$$\begin{aligned} N_{\text{vert}} &\leq \sum_{1 \leq \ell < H^{\frac{2}{\beta}}} \frac{c}{\log H} \left(1 + \log\left(\frac{H^{\frac{2}{\beta}}}{\ell}\right)\right) + \sum_{H^{\frac{2}{\beta}} \leq \ell \leq |W|} \frac{c}{\log H} H \ell^{-\frac{\beta}{2}} \\ &\leq c H^{\frac{2}{\beta}}/\log H + c H^{\frac{2}{\beta}}(1 + 1\{\beta = 2\} \log(L/H^{\frac{2}{\beta}}))/\log H, \end{aligned} \quad (5.28)$$

whence (5.25).

When $\beta > \alpha > 1 + \frac{\beta}{2}$, (so that $\nu > 1$ and $\beta > 2$), the last line of (5.13) yields that:

$$N_{\text{vert}} \leq \sum_{\ell=1}^{|W|} c \left(1 \wedge \left(\frac{H^{\frac{2}{\beta}}}{\ell}\right)^{\nu}\right) \leq c H^{\frac{2}{\beta}} + c H^{\frac{2\nu}{\beta}} \sum_{\ell \geq H^{\frac{2}{\beta}}} \ell^{-\nu} \leq c H^{\frac{2}{\beta}}, \quad (5.29)$$

whence (5.25).

Finally when $\beta > \alpha = 1 + \frac{\beta}{2}$, (so that $\nu = 1$ and $\beta > 2$), the last line of (5.13) now yields that:

$$N_{\text{vert}} \leq \sum_{\ell=1}^{|W|} c \left(1 \wedge \left(\frac{H^{\frac{2}{\beta}}}{\ell}\right)^{\nu}\right) \leq c H^{\frac{2}{\beta}}(1 + \log(L/H^{\frac{2}{\beta}})), \quad (5.30)$$

whence (5.25).

We then turn to the proof of (5.26). When $\alpha > 1 + \frac{\beta}{2}$, then $0 > \frac{2}{\beta}(1-\nu) = \frac{2}{\beta}(1 + \frac{\beta}{2} - \alpha) = 1 - \frac{2}{\beta}(\alpha - 1)$, and the first line of (5.14) implies that:

$$N_{\text{hor}} \leq \sum_{k=1}^H c k^{\frac{2}{\beta}(1-\nu)} \leq c H^{(1+\frac{2}{\beta}(1-\nu))+} + c 1\{\alpha = 1 + \beta\} \log H, \quad (5.31)$$

whence (5.26).

Finally when $\alpha = 1 + \frac{\beta}{2}$, then $\nu = 1$ and the second line of (5.14) yields that

$$\begin{aligned} N_{\text{hor}} &\leq \sum_{k=1}^H \frac{c}{\log L} \left(1 + \log \left(\frac{L^{\frac{\beta}{2}}}{k} \right) \right) \\ &\leq c H / \log L + c H \log(L^{\frac{\beta}{2}} / H) / \log L, \end{aligned} \quad (5.32)$$

whence (5.26). This concludes the proof of Lemma 5.3. \square

We are now ready to prove (5.9). For this purpose we bound A in (5.11) with the help of Khasminskii's Lemma, cf. [16], (2.46) of [10], or [8], p. 71. We thus find that for all rectangles $D = W \times J$ as stated above Lemma 5.3 we have:

$$\sup_{x \in E} E_x \left[\exp \left\{ \frac{c}{N_{\text{vert}}} \sum_{\bar{y} \in W} 1 \{ H_{J_{\bar{y}}} < \infty \} \right\} \right] \leq 2, \quad \text{and} \quad (5.33)$$

$$\sup_{x \in E} E_x \left[\exp \left\{ \frac{c}{N_{\text{hor}}} \sum_{\bar{z} \in J} 1 \{ H_{W_{\bar{z}}} < \infty \} \right\} \right] \leq 2. \quad (5.34)$$

As a result of the exponential Chebyshev inequality and (5.11) we thus find:

$$A \leq c N^{\frac{\beta}{2}} 2^M \left[\exp \left\{ -c \frac{L}{N_{\text{vert}}} \right\} + \exp \left\{ -c \frac{H}{N_{\text{hor}}} \right\} \right]. \quad (5.35)$$

However from (5.10) and Lemma 5.3 we find that for $L \geq c$,

$$\frac{L}{N_{\text{vert}}} \geq c \frac{L}{H^{\frac{2}{\beta}}} \frac{(1 + 1 \{ \alpha = \beta \} \log L)}{1 + \log(L / H^{\frac{2}{\beta}})} \geq c \frac{\log L}{\log \log L}, \quad \text{and} \quad (5.36)$$

$$\frac{H}{N_{\text{hor}}} \geq c \frac{\log L}{1 + \log(L / H^{\frac{2}{\beta}})} \geq c \frac{\log L}{\log \log L}. \quad (5.37)$$

Inserting these bounds in (5.35), (recall that $N = \log L$), implies that A tends to zero as L goes to infinity. This proves (5.9) and thus concludes the proof of Theorem 5.1. \square

Remark 5.4 In the above proof the methods of Sect. 3 make the case where $\nu > \beta$, i.e. $\alpha > \frac{3}{2}\beta$, much simpler to treat. Indeed M in (5.8) vanishes and so does the expression under the supremum in (5.9). This immediately yields the claim (5.5).

In the case $\alpha \leq \frac{3}{2}\beta$, the choice of rectangles D where the height H is such that $H^{\frac{2}{\beta}}$ is slightly smaller (by a logarithmic factor) than the base $|W| \in [L, 5L]$, see (5.10) and below (5.11), is truly useful when handling the situation where $\beta > \alpha$, (and $v \geq 1$), as can be seen from (5.29), (5.30). Such a procedure works because N_{hor} nevertheless remains “negligible” with respect to H , as can be seen in (5.31), (5.32).

We state a consequence of (5.4) in terms of the size of the finite connected components of \mathcal{V}^u in half-planes when $u < \tilde{u}$. The notation is the same as in Theorem 5.1, with $0 < \tilde{\kappa}(u) < 1$ introduced in (5.4).

Corollary 5.5 *When $u < \tilde{u}$, there exist positive $c_8(u)$, $c_9(u)$ such that*

$$\sup_{\mathcal{P}, x \in \mathcal{P}} \mathbb{P}[\text{the connected component of } x \text{ in } \mathcal{V}^u \cap \mathcal{P} \text{ is finite and intersects } \partial_{\text{int}} B(x, L)] \leq c_8 e^{-c_9 L^{\tilde{\kappa}}}, \quad \text{for } L \geq 1. \quad (5.38)$$

Proof On the event inside the probability one can find a $*$ -path in $\mathcal{I}^u \cap \mathcal{P}$ separating in \mathcal{P} the connected component of x in $\mathcal{V}^u \cap \mathcal{P}$ from infinity. The respective G - and \mathbb{Z} -projections of the range of the $*$ -path are “intervals” containing the respective G - and \mathbb{Z} -projections of x , and either the G -projection contains at least L points or the \mathbb{Z} -projection contains at least $L^{\frac{\beta}{2}}$ points. One can then consider the rightmost point of the $*$ -path having same height as x . A variation of the argument used to bound from below the last factor of (5.7) combined with the inequalities $\sum_{k \geq 0} e^{-c(u)(2^k L)^{\tilde{\kappa}}} \leq \sum_{k \geq 0} e^{-c'(u)(k+1)L^{\tilde{\kappa}}} \leq c(u) e^{-c'(u)L^{\tilde{\kappa}}}$ yields the claim (5.38) in a straightforward fashion. \square

Remark 5.6

- (1) In the important special case $E = \mathbb{Z}^{d+1}$, $d \geq 2$, when the sup-norm distance $d_{\infty}(\cdot, \cdot)$ replaces $d(\cdot, \cdot)$, the results of Theorem 5.1 and Corollary 5.5, show that the plane exponent:

$$\tilde{u}_{\text{pl}} = \inf \left\{ u \geq 0; \lim_{L \rightarrow \infty} \mathbb{P}[B_{\infty}(0, L) \overset{*-\mathcal{I}^u \cap \mathbb{Z}^2}{\longleftrightarrow} B(0, 2L)] > 0 \right\}, \quad (5.39)$$

(which coincides with \tilde{u}), is such that for any $d \geq 2$,

$$0 < \tilde{u}_{\text{pl}} \leq u_*, \quad (5.40)$$

that for $u < \tilde{u}_{\text{pl}}$,

$$\mathbb{P}\text{-a.s., } \mathcal{V}^u \text{ percolates in all planes of } \mathbb{Z}^d, \quad (5.41)$$

and there exist $c(u)$, $c'(u)$, $0 < \gamma(u) < 1$, such that for $L \geq 1$,

$$\mathbb{P}[B_\infty(0, L) \stackrel{*}{\longleftrightarrow}^{\mathcal{I}^u \cap \mathbb{Z}^2} \partial_{\text{int}} B_\infty(0, 2L)] \leq c e^{-c' L^\gamma}, \quad \text{and} \quad (5.42)$$

$$\mathbb{P}[\text{the connected component of } 0 \text{ in } \mathcal{V}^u \cap \mathbb{Z}^2 \text{ is finite and intersects } \partial_{\text{int}} B_\infty(0, L)] \leq c e^{-c' L^\gamma}. \quad (5.43)$$

The above sharpens previously known results of [18], cf. Remark 3.5 (2), where arbitrary polynomial decay in L for the left-hand side of (5.42), when u is sufficiently small has been established. It also complements the separation results from [26], for small $u > 0$ and $d + 1 \geq 5$, see also (3.19) and Remark 3.8 (1).

- (2) It is a very interesting question to understand what happens when $\alpha < 1 + \frac{\beta}{2}$. Does one of the values \tilde{u} or u_* vanish, or both? Can one successfully apply Corollary 3.7 to a cascading family in the spirit of the family \mathcal{S} of separation events, which has been introduced in Remark 3.3 (3)? The special case of $E = G \times \mathbb{Z}$, where G is the discrete skeleton of the Sierpinski gasket endowed with its natural weights, corresponds to $\alpha = \frac{\log 3}{\log 2}$, $\beta = \frac{\log 5}{\log 2}$, cf. [3, 15], and is somehow emblematic of this puzzle.

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Appendix A

The main object of this appendix is to prove that the collection \mathcal{S} of “separation events” introduced in Remark 3.3 3) has the cascading property. The proof is in essence an adaptation of the arguments that appear in Lemma 5.1 and Theorem 5.2 of Teixeira [27], but the geometry is possibly quite different in the present set-up. Our main result is stated in Proposition A.2. We refer to the beginning of Sect. 1 for notation, and recall that $d_E(\cdot, \cdot)$ denotes the graph-distance on E , which should not be confused with $d(\cdot, \cdot)$, see (0.3).

We first note that for any subsets A, B of E , one has

$$d_E(A, B) > 1 \iff \overline{A} \cap B = \emptyset \iff A \cap \overline{B} = \emptyset. \quad (\text{A.1})$$

We say that A and B are separated by C in U , all of them being subsets of E , when $d_E(A, B) > 1$ and any path from ∂A to ∂B remaining in U meets C . The collection of events $\mathcal{S} = (S_{x,L})_{x \in E, L \geq 1 \text{ integer}}$ introduced in (3.20) and

discussed in this appendix is defined by:

$$S_{x,L} = \left\{ \sigma \in \{0, 1\}^E; \text{there exist connected subsets } A_1 \text{ and } A_2 \text{ of } \overline{B(x, 3L)} \right. \\ \left. \text{with } d(\cdot, \cdot)\text{-diameter at least } L, \text{ separated by } \Sigma(\sigma) \text{ in } B(x, 5L) \right\}, \quad (\text{A.2})$$

where $\Sigma(\sigma) = \{x \in E; \sigma(x) = 1\}$. For simplicity we will often write Σ in place of $\Sigma(\sigma)$, and diameter in place of $d(\cdot, \cdot)$ -diameter. Our first result is:

Lemma A.1 (ℓ multiple of 100, $L \geq 1$ integer, x_* in E)

Assume $A_1, A_2 \subseteq \overline{B(x_*, 3\ell L)}$ are connected subsets of diameter at least ℓL , that $\sigma \in \{0, 1\}^E$ is such that Σ separates A_1 and A_2 in $B(x_*, 5\ell L)$, and that x_1, \dots, x_M is a sequence in E such that:

$$\begin{aligned} & \text{(i)} \quad B(x_i, 5L) \subseteq B(x_*, 5\ell L), \quad 1 \leq i \leq M, \\ & \text{(ii)} \quad d(x_i, x_{i+1}) \leq L, \quad \text{for } 1 \leq i < M, \\ & \text{(iii)} \quad A_1 \cap B(x_1, 2L) \neq \emptyset \quad \text{and} \quad A_2 \cap B(x_M, 2L) \neq \emptyset. \end{aligned} \quad (\text{A.3})$$

It then follows that

$$\text{for some } i \in \{1, \dots, M\}, \quad \sigma \in S_{x_i, L}. \quad (\text{A.4})$$

Proof Let \mathcal{C} denote the connected component of A_1 in $(A_1 \cup \Sigma^c) \cap B(x_*, 5\ell L)$. By construction we have:

$$\partial \mathcal{C} \cap B(x_*, 5\ell L) \subseteq \Sigma. \quad (\text{A.5})$$

Moreover we also have

$$\overline{A_2} \cap \mathcal{C} = \emptyset. \quad (\text{A.6})$$

Indeed otherwise we could find a path in $B(x_*, 5\ell L)$ from A_1 to $\overline{A_2}$ not meeting Σ after its last visit to A_1 , and as a result we could find a path from ∂A_1 to ∂A_2 in $B(x_*, 5\ell L)$ not meeting Σ , a contradiction.

By (A.6) and (A.5) we now see that $d_E(\mathcal{C}, A_2) > 1$, and any path from $\partial \mathcal{C}$ to ∂A_2 in $B(x_*, 5\ell L)$ meets Σ , i.e.,

$$\mathcal{C} \text{ and } A_2 \text{ are separated by } \Sigma \text{ in } B(x_*, 5\ell L). \quad (\text{A.7})$$

By (A.3) (iii) we know that $\mathcal{C} \cap B(x_1, 2L) \neq \emptyset$, and we denote by i_* the largest $1 \leq i \leq M$, such that $\mathcal{C} \cap B(x_i, 2L) \neq \emptyset$. We first note that

$$\text{when } i_* = M, \text{ then } \sigma \in S_{x_M, L}. \quad (\text{A.8})$$

Indeed \mathcal{C} and A_2 have diameter at least $\ell L > 6L + 2$ and thus both meet $\partial B(x_M, 3L)$ and $B(x_M, 2L)$. We can hence choose connected subsets A'_1 of $\mathcal{C} \cap \overline{B(x_M, 3L)}$ and A'_2 of $A_2 \cap \overline{B(x_M, 3L)}$ with diameter at least L . Due to (A.7) they are separated by Σ in $B(x_*, 5\ell L)$ and hence in $B(x_M, 5L)$. The claim (A.8) now follows.

We then observe that

$$\text{when } 1 \leq i_* < M, \quad \text{then } \sigma \in S_{x_{i_*}, L}. \quad (\text{A.9})$$

Indeed consider the “vertical segment” (with π_G and $\pi_{\mathbb{Z}}$ the respective G -projection and \mathbb{Z} -projection on E):

$$\begin{aligned} J &= \{x \in E; \pi_G(x) = \pi_G(x_{i_*+1}) \text{ and } -1 \leq \pi_{\mathbb{Z}}(x) - \pi_{\mathbb{Z}}(x_{i_*+1}) \leq L^{\beta/2}\} \\ &\subseteq B(x_{i_*+1}, L). \end{aligned}$$

By assumption $\mathcal{C} \cap B(x_{i_*+1}, 2L) = \emptyset$, so that $\overline{J} \cap \mathcal{C} \subseteq B(x_{i_*+1}, 2L) \cap \mathcal{C} = \emptyset$, and $d_E(J, \mathcal{C}) > 1$. Suppose that $\sigma \notin S_{x_{i_*}, L}$. Due to (A.3) ii) we see that $J \subseteq \overline{B(x_{i_*}, 3L)}$, and we can extract a connected subset \mathcal{C}' of $\mathcal{C} \cap \overline{B(x_{i_*}, 3L)}$ with diameter at least L and a path from $\partial \mathcal{C}'$ to ∂J in $B(x_{i_*}, 5L) \setminus \Sigma$. This shows that \mathcal{C} extends to ∂J and thus meets $B(x_{i_*+1}, 2L)$, a contradiction. The claim (A.4) now follows from (A.8), (A.9), and Lemma A.1 is proved. \square

We now come to the main result of this appendix.

Proposition A.2

\mathcal{S} is a family of increasing events on $\{0, 1\}^E$ that cascades with complexity at most $\alpha + \frac{\beta}{2}$. (A.10)

Proof Clearly \mathcal{S} is a family of increasing events which satisfy (3.1). For any given ℓ multiple of 100, x_* in E , and integer $L \geq 1$, we want to find a finite subset Λ of E for which (3.2)–(3.5) hold, with $\lambda = \alpha + \frac{\beta}{2}$ and x_* in place of x in (3.2)–(3.5). Without loss of generality and to simplify notation, we assume that $x_* = (y_*, 0)$.

We consider $\Lambda_G \subseteq B_G(y_*, 5\ell L)$ defined as follows. When $L \leq 4$, $\Lambda_G = B_G(y_*, 5\ell L)$ and when $L > 4$, Λ_G is a maximal set of points in $B_G(y_*, 5\ell L)$ with mutual d_G -distance bigger than $L/4$. By construction we see that

$$B_G(y_*, 5\ell L) \subseteq \bigcup_{y \in \Lambda_G} B(y, L/4) \quad (\text{A.11})$$

and by the α -Ahlfors regularity of G , see (1.7), with similar arguments as in the proof of Proposition 3.2, we find that

$$|\Lambda_G| \leq c \ell^\alpha. \quad (\text{A.12})$$

Further note that for $y \in G$, $R \geq 0$, the ball $B_G(y, R)$ is connected, and using (A.11) when $L > 4$, we have the following local connectivity property of Λ_G :

$$\begin{aligned} &\text{when } B_G(y, R) \subseteq B_G(y_*, 5\ell L) \text{ and } y', y'' \in \Lambda_G \cap B_G(y, R), \text{ there is a} \\ &\text{sequence } y_1, \dots, y_M \text{ in } \Lambda_G \cap B_G(y, R + \frac{L}{4}) \text{ such that } y_1 = y', y_M = y'', \\ &\text{and } d_G(y_i, y_{i+1}) \leq L, \text{ for } 1 \leq i < M. \end{aligned} \quad (\text{A.13})$$

We also introduce the finite subset of \mathbb{Z} :

$$\Lambda_{\mathbb{Z}} = a \mathbb{Z} \cap [-(5\ell L)^{\beta/2}, (5\ell L)^{\beta/2}], \quad \text{where } a = \max(1, \lfloor (L/4)^{\frac{\beta}{2}} \rfloor). \quad (\text{A.14})$$

We then define $\Lambda = \Lambda_G \times \Lambda_{\mathbb{Z}}$, and note that

$$\begin{aligned} \Lambda &\subseteq B(x_*, 5\ell L), \text{ and } B(x_*, 5\ell L) \subseteq \bigcup_{x \in \Lambda} B\left(x, \frac{L}{4}\right), \\ &\text{(when } L \leq 4, \Lambda \text{ actually coincides with } B(x_*, 5\ell L)). \end{aligned} \quad (\text{A.15})$$

Further one has

$$|\Lambda| \leq c \ell^{\alpha + \frac{\beta}{2}}. \quad (\text{A.16})$$

Thus Λ satisfies (3.2), (3.3), and Proposition A.2 will follow once we prove (3.4).

To this end we consider $\sigma \in S_{x_*, 5\ell L}$ and A_1, A_2 connected subsets of $\overline{B(x_*, 3\ell L)}$ with diameter at least ℓL separated by Σ in $B(x_*, 5\ell L)$, see below (A.2) for notation.

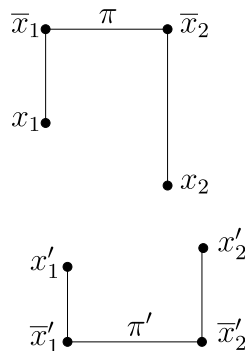
By Lemma A.1, the claim (3.4) will follow once we find two sequences in Λ , $\pi = (\pi(i), 1 \leq i \leq M)$ and $\pi' = (\pi'(j), 1 \leq j \leq M')$, such that

$$\begin{aligned} &\text{(i) both } \pi \text{ and } \pi' \text{ satisfy (A.3).} \\ &\text{(ii) } d(\pi(i), \pi'(j)) \geq \frac{\ell}{100} L, \quad 1 \leq i \leq M, \quad 1 \leq j \leq M'. \end{aligned} \quad (\text{A.17})$$

We write $\tilde{\ell} = \ell/100 (\geq 1 \text{ by assumption})$. As we now explain, we can find x_1, x'_1, x_2, x'_2 in Λ so that

$$\begin{aligned} &\text{(i) the four points have mutual distance at least } 24\tilde{\ell} L. \\ &\text{(ii) } A_1 \cap B(x_1, L/4) \neq \emptyset, \quad A_1 \cap B(x'_1, L/4) \neq \emptyset, \\ &\text{(iii) } A_2 \cap B(x_2, L/4) \neq \emptyset, \quad A_2 \cap B(x'_2, L/4) \neq \emptyset. \end{aligned} \quad (\text{A.18})$$

Fig. 3 A schematic representation of π and π' in case (1)



Indeed we can pick two points of A_1 with mutual distance at least ℓL . Since A_2 is connected and has diameter at least ℓL , it has at least one point outside the union of the two balls of radius $50\tilde{\ell}L - 1$ centered at the above two points. Likewise A_2 must exit the ball of radius $25\tilde{\ell}L$ centered at this last point and hence meet the interior boundary of this ball. In this fashion we have constructed two points on A_1 and two points on A_2 , so that these four points have mutual distance at least $25\tilde{\ell}L - 1$. By (A.15) we can thus find $x_1 = (y_1, z_1)$, $x'_1 = (y'_1, z'_1)$, $x_2 = (y_2, z_2)$, $x'_2 = (y'_2, z'_2)$ in Λ satisfying (A.18).

Up to relabelling we assume that $z_1 \geq z'_1$ and $z_2 \geq z'_2$. We will construct the sequences π and π' satisfying (A.17) when $z_1 \geq z_2$. The case where $z_1 < z_2$ is handled in a similar fashion exchanging the role of 1 and 2 in what follows. We from now on assume $z_1 \geq z_2$, and treat separately the case where z_2 lies in the interval $[z'_1, z_1]$ and the case where it does not.

(1) $z'_1 \leq z_2 \leq z_1$:

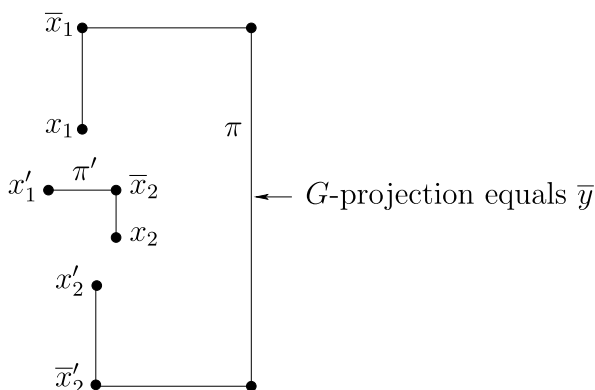
We construct π and π' as follows. We pick $\bar{x}_1 = (y_1, \bar{z}_1)$ having same G -projection as x_1 , with \bar{z}_1 the largest elements of $\Lambda_{\mathbb{Z}}$ smaller or equal to $z_1 + (25\tilde{\ell}L)^{\beta/2}$, and define $\bar{x}_2 = (y_2, \bar{z}_1)$. By (A.13) we can find a sequence in $\Lambda_G \cap B_G(y_*, 3\ell L + 1 + \frac{L}{4})$ linking y_1 and y_2 making steps of d_G -distance at most L . We thus define π fulfilling (A.3) which first moves up from x_1 to \bar{x}_1 , then horizontally from \bar{x}_1 to \bar{x}_2 , using the above mentioned sequence, and then down from \bar{x}_2 to x_2 .

Likewise to construct π' we consider $\bar{x}'_1 = (y'_1, \bar{z}'_1)$ and $\bar{x}'_2 = (y'_2, \bar{z}'_1)$, where \bar{z}'_1 denotes the smallest element of $\Lambda_{\mathbb{Z}}$ bigger or equal to $z'_1 \wedge z'_2 - (25\tilde{\ell}L)^{\beta/2}$. We link y'_1 and y'_2 by a sequence in $\Lambda_G \cap B_G(y, 3\ell L + 1 + \frac{L}{4})$ as above, and construct π' satisfying (A.3), which first moves down from x'_1 to \bar{x}'_1 , then horizontally from \bar{x}'_1 to \bar{x}'_2 , and then vertically from \bar{x}'_2 to x'_2 .

The above constructed π and π' , see Fig. 3, thus satisfy (A.3) and

$$d(\pi(i), \pi(j)) \geq 24\tilde{\ell}L, \quad \text{for } 1 \leq i \leq M, 1 \leq j \leq M'. \quad (\text{A.19})$$

Fig. 4 A schematic representation of π, π' in case 2(a)



(2) $z_1 \geq z'_1 > z_2 (\geq z'_2)$:

We consider two sub-cases.

(a) $d_G(y'_1, y_2) \leq 11\tilde{\ell}L$.

We introduce $\bar{x}_2 = (y_2, z'_1)$ and link y'_1 and y_2 in $\Lambda_G \cap B_G(y'_1, 11\tilde{\ell}L + \frac{L}{4})$ by a sequence making steps of d_G -distance at most L . We then construct π' which moves first horizontally from x'_1 to \bar{x}_2 , using the above sequence and then downwards from \bar{x}_2 to x'_2 .

To construct π we introduce \bar{x}_1 and \bar{x}'_2 in Λ with same G -projection as x_1 and x'_2 respectively, with \bar{x}_1 the “highest point” in Λ with distance at most $25\tilde{\ell}L$ from x_1 and \bar{x}'_2 the “lowest point” in Λ with distance at most $25\tilde{\ell}L$ from x'_2 , see Fig. 4.

We then pick \bar{y} in $\Lambda_G \cap \overline{B_G(y_*, 3\ell L)}$ having at least d_G -distance $50\tilde{\ell}L$ from $B_G(y'_1, 11\tilde{\ell}L + \frac{L}{4})$. This is possible because $\overline{B_G(y_*, 2\ell L)}$ has d_G -diameter at least $2\ell L$ and it suffices to pick $y \in \overline{B_G(y_*, 2\ell L)}$ with $d_G(y, y'_1) \geq \ell L$ and then \bar{y} in Λ_G within d_G -distance $\frac{L}{4}$ from y .

Then π is constructed as follows. It first moves upwards from x_1 to \bar{x}_1 then horizontally to the point having G -projection \bar{y} , then downwards to the point with same \mathbb{Z} -projection as \bar{x}'_2 , then horizontally to \bar{x}'_2 , and then upwards to x'_2 . This construction can be performed so that the G -projection of π lies in $\Lambda_G \cap B(y_*, 3\ell L + 1 + \frac{L}{4})$.

The above constructed paths π, π' satisfy (A.3) and

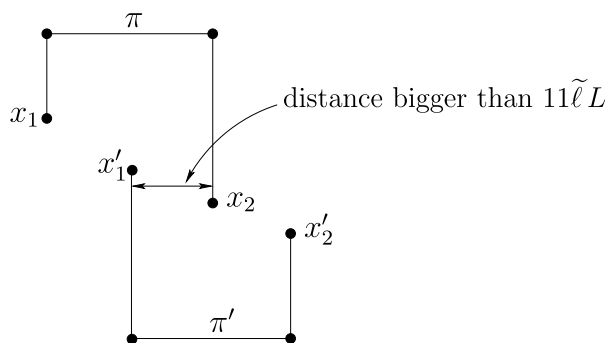
$$d(\pi(i), \pi(j)) \geq 12\tilde{\ell}L. \quad (\text{A.20})$$

The remaining sub-case of (2) corresponds to

(b) $d_G(y'_1, y_2) > 11\tilde{\ell}L$.

We then construct π linking x_1 and x_2 and π' linking x'_1 and x'_2 as in (1).

Fig. 5 A schematic representation of π and π' in case 2(b)



The paths π and π' constructed in this fashion, see Fig. 5, satisfy (A.3) and

$$d(\pi(i), \pi'(j)) \geq 11\tilde{\ell}L, \quad 1 \leq i \leq M, \quad 1 \leq j \leq M'. \quad (\text{A.21})$$

Combining (A.19)–(A.21), we have thus completed the construction of π, π' satisfying (A.17) and Proposition A.2 follows. \square

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